A CLASS OF ADDING MACHINE AND JULIA SETS

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ABSTRACT. In this work we define a stochastic adding machine associated to the Fibonacci base and to a probabilities sequence $\overline{p} = (p_i)_{i\geqslant 1}$. We obtain a Markov chain whose states are the set of nonnegative integers. We study probabilistic properties of this chain, such as transience and recurrence. We also prove that the spectrum associated to this Markov chain is connected to the fibered Julia sets for a class of endomorphisms in $\mathbb{C}^2$.

CONTENTS

INTRODUCTION

The goal of this work is to study a problem in the areas of dynamical systems, probability and spectral operator theory. These problems involve stochastic adding machine, spectrum of transition operator associated to Markov chains and fibered Julia sets of endomorphisms in $\mathbb{C}^2$.

Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic map. The filled Julia set associated to g is by definition the set $K(g) = \{z \in \mathbb{C} : g^n(z) \text{ is bounded}\}$, where g^n is the $n-$th iterate of g.

A connection between Julia sets in $\mathbb{C}$ and stochastic adding machine was given by Killeen and Taylor in [KT1]. They defined the stochastic adding machine as follows: let $N \in \mathbb{N} = \{0, 1, 2, \ldots\}$ be a nonnegative integer. By using the greedy algorithm we may write N as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N)2^i$, in a unique way, where $\varepsilon_i(N) \in \{0, 1\}$, for all $i \in \{0, \ldots, k(N)\}$. So, the representation of N in base 2 is given by $N = \varepsilon_{k(N)}(N) \ldots \varepsilon_0(N)$. They defined a systems of evolving equation that calculates the digits of $N + 1$ in base 2, introducing an auxiliary variable "carry 1", $c_i(N)$, for each digit $\varepsilon_i(N)$, as follows:

Date: June 21, 2017.

2010 *Mathematics Subject Classification.* Primary 37A30, 37F50; Secondary 60J10, 47A10.

Key words and phrases. Markov chains; stochastic adding machines; spectrum of transition operator; Julia sets; fibered Julia sets.

Define $c_{-1}(N+1) = 1$ and for all $i \geqslant 0$, do

$$\begin{aligned} \varepsilon_i(N+1) &= (\varepsilon_i(N) + c_{i-1}(N+1)) \mod 2; \\ c_i(N+1) &= \left[\frac{\varepsilon_i(N) + c_{i-1}(N+1)}{2}\right], \end{aligned} \tag{0.1}$$

where $[x]$ is the integer part of $x \in \mathbb{R}^+$.

Killeen and Taylor [KT1] defined a stochastic adding machine considering a family of independent, identically distributed randon variables $\{e_i(n) : i \geqslant 0,\ n \in \mathbb{N}\}$, parametrized by nonnegative integers i and n, where each $e_i(n)$ takes the value 0 with probability $1-p$ and the value 1 with probability p. More precisely, they defined a stochastic adding machine as follows: let N be a nonnegative integer and consider the sequences $(\varepsilon(N+1))_{i\geqslant 0}$ and $(c_i(N+1))_{i\geqslant -1}$ defined by $c_{-1}(N+1) = 1$ and for all $i \geqslant 0$

$$\begin{aligned} \varepsilon_i(N+1) &= (\varepsilon_i(N) + e_i(N)c_{i-1}(N+1)) \mod 2; \\ c_i(N+1) &= \left[\frac{\varepsilon_i(N) + e_i(N)c_{i-1}(N+1)}{2}\right]. \end{aligned} \tag{0.2}$$

Killeen and Taylor [KT1] studied the spectrum of the transition operator S associated to the stochastic adding machine in base 2, acting in $l^\infty(\mathbb{N})$, and they proved that the spectrum of S in $l^\infty(\mathbb{N})$ is equal to the filled Julia set of the quadratic map $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = \left(\frac{z-(1-p)}{p}\right)^2$, i.e. $\sigma(S) = \{z \in \mathbb{C} : (f^n(z))_{n\geqslant 0} \text{ is bounded}\}$.

In [MSV] and [MV], the authors considered the stochastic adding machine taking a probabilities sequence $\overline{p} = (p_i)_{i\geqslant 1}$, where the probability change in each state, i.e. on the description (0.2) we have $e_i(N) = 1$ with probability p_{i+1} and $e_i(N) = 0$ with probability $1 - p_{i+1}$, for all $i \geqslant 0$, and they constructed the transition operator $S_{\overline{p}}$ related to this probabilities sequence.

In particular, in [MSV] they proved that the Markov chain is null recurrent if only if $\prod_{i=1}^{+\infty} p_i = 0$. Otherwise the chain is transient. Furthermore, they proved that the spectrum of $S_{\overline{p}}$ acting in c_0, c and l^α, $1 \leqslant \alpha \leqslant \infty$, is equal to the fibered Julia set $E_{\overline{p}} := \left\{z \in \mathbb{C} : (\tilde{f}_j(z)) \text{ is bounded}\right\}$, where $\tilde{f}_j := f_j \circ \ldots \circ f_1$ and $f_j : \mathbb{C} \longrightarrow \mathbb{C}$ are maps defined by $f_j(z) = \left(\frac{z-(1-p_j)}{p_j}\right)^2$, for all $j \geqslant 1$. Moreover, the spectrum of $S_{\overline{p}}$ in $l^\infty(\mathbb{N})$ is equal to the point spectrum.

In this paper, instead of base 2, we will consider the Fibonacci base $(F_n)_{n\geqslant 0}$ defined by $F_n = F_{n-1} + F_{n-2}$, for all $n \geqslant 2$, where $F_0 = 1$ and $F_1 = 2$. Also, we will consider a probabilities sequence $\overline{p} = (p_i)_{i\geqslant 1}$, instead of an unique probability p (as was done in [MS] and [MU], for a large class of recurrent sequences of order 2). Thenceforth, we will define the Fibonacci stochastic adding machine and considering the transition operator S, we will prove that the Markov chain is transient if only if $\prod_{i=1}^{\infty} p_i > 0$. Otherwise, if $\sum_{i=1}^{+\infty} p_i = +\infty$ then the Markov chains is null recurrent and if $\sum_{i=2}^{+\infty} p_i F_{2(i-1)} < +\infty$ then the Markov chain is positive recurrent.

We will compute the point spectrum and prove that it is connected to the fibered Julia sets for a class of endomorphisms in $\mathbb{C}^2$. Precisely, $\sigma_{pt}(S) \subset E \subset \sigma_a(S)$ where $E = \{z \in \mathbb{C} : (g_n \circ \ldots \circ g_0(z,z))_{n\geqslant 0} \text{ is bounded}\}$ and $g_n : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are maps defined by

$g_0(x,y) = \left(\frac{x-(1-p_1)}{p_1}, \frac{y-(1-p_1)}{p_1}\right)$ and $g_n(x,y) = \left(\frac{1}{r_n}xy - \left(\frac{1}{r_n}-1\right), x\right)$ for all $n \geqslant 1$, where $r_n = p_{\left[\frac{n+1}{2}\right]+1}$. Moreover, if $\liminf_{i\to+\infty} p_i > 0$ then E is compact and $\mathbb{C}\setminus E$ is connected.

The paper is organized as follows. In subsection 1.1 we give the definition of Fibonacci adding machine and explain how this machine is defined by a finite transductor. In subsection 1.2 we define the stochastic Fibonacci adding machine and we obtain the transition operator of the Markov chain. In section 2 we give a necessary and sufficient condition for transience, a sufficient condition for null recurrence and for positive recurrence. In section 3 is devoted to provide an exact description of the spectra of these transition operators acting on $l^\infty(\mathbb{N})$. The section 4 contains results about connectedness properties of the fibered Julia sets.

1. Stochastic Fibonacci adding machine

1.1. **Adding machine.** Let $(F_i)_{i\geqslant 0}$ be the Fibonacci sequence defined by $F_0 = 1$, $F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$, for all $n \geqslant 2$. By using the greedy algorithm we can write every nonnegative integer number N, in a unique way, as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N)F_i$, where $\varepsilon_i(N) = 0$ or 1 and $\varepsilon_{i+1}(N)\varepsilon_i(N) \neq 11$, for all $0 \leqslant i < k(N)$.

Example 1.1. *a)* $12 = 8+3+1 = F_4+F_2+F_0 = 10101$; *b)* $14 = 13+1 = F_5+F_1 = 100010$.

A way to calculate the digits of $N+1$ in Fibonacci base is given by a finite transducer $\mathcal{T}$ on $A^* \times A^*$, where $A = \{0,1\}$ is a finite alphabet and A^* is the set of finite words on A. The transducer $\mathcal{T}$ of the Fibonacci adding machine, represented on figure 1, is formed by two states, an initial state I and a terminal state T. The initial state is connected to itself by two arrows. One of them is labelled by (10/00) and the other by (1/0). There is also one arrow going from the initial state to the terminal one. This arrow is labelled by (00/01). The terminal state is connected to itself by two arrows. One of them is labelled by (0/0) and the other by (1/1).

Remark 1.2. The transducer $\mathcal{T}$ was defined in [MU].

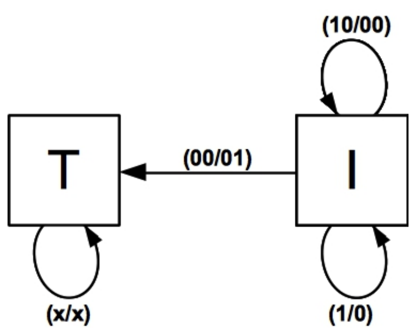


FIGURE 1. Transductor of adding machine in Fibonacci base.

Example 1.3. If $N = 17 = 100101$ then N corresponds to the path

$$(T, 1/1, T)(T, 00/01, I)(I, 10/00, I)(I, 1/0, I).$$

Hence, $N+1 = 1010000 = 18$.

Like was defined by Killeen and Taylor in description (0.1), we may construct an algorithm that computes the digits of $N+1$. This algorithm can be described by introducing an auxiliary binary "carry" variable $c_i(N+1)$ for each digit $\varepsilon_i(N)$ by the following manner:

Put $c_{-1}(N+1)=1$ and

i) If $\varepsilon_0(N)=0$ then for all $i\geqslant 0$, do:

$$\begin{aligned} \varepsilon_{2i}(N+1) &= \left[\frac{\varepsilon_{2i}(N)+c_{i-1}(N+1)}{\varepsilon_{2i+1}(N)c_{i-1}(N+1)+1}\right]; \\ \varepsilon_{2i+1}(N+1) &= \left[\frac{\varepsilon_{2i+1}(N)}{c_{i-1}(N+1)+1}\right]; \\ c_i(N+1) &= c_{i-1}(N+1)\varepsilon_{2i+1}(N). \end{aligned} \tag{1.1}$$

ii) If $\varepsilon_0(N)=1$ then put $\varepsilon_0(N+1)=0$, $c_0(N+1)=1$ and, for all $i\geqslant 1$, do:

$$\begin{aligned} \varepsilon_{2i-1}(N+1) &= \left[\frac{\varepsilon_{2i-1}(N)+c_{i-1}(N+1)}{\varepsilon_{2i}(N)c_{i-1}(N+1)+1}\right]; \\ \varepsilon_{2i}(N+1) &= \left[\frac{\varepsilon_{2i}(N)}{c_{i-1}(N+1)+1}\right]; \\ c_i(N+1) &= c_{i-1}(N+1)\varepsilon_{2i}(N). \end{aligned} \tag{1.2}$$

Remark 1.4. If $c_{j-1}(N+1)=0$ for some nonnegative integer $j\geqslant 1$ then $c_i(N+1)=0$ for all $i\geqslant j-1$. Moreover, in description (1.1) we have $\varepsilon_l(N+1)=\varepsilon_l(N)$ for all $l\geqslant 2j$ and in description (1.2) we have $\varepsilon_l(N+1)=\varepsilon_l(N)$ for all $k\geqslant 2j-1$.

Example 1.5. *a*) Let $N=3=0100=\varepsilon_3(3)\varepsilon_2(3)\varepsilon_1(3)\varepsilon_0(3)$. By description (1.1) we have $c_{-1}(4)=1$ and

$$\begin{cases} \varepsilon_0(4)=\left[\frac{\varepsilon_0(3)+c_{-1}(4)}{\varepsilon_1(3)c_{-1}(4)+1}\right]=1; \\ \varepsilon_1(4)\left[\frac{\varepsilon_1(3)}{c_{-1}(4)+1}\right]=0; \\ c_0(4)=c_{-1}(4)\varepsilon_1(3)=0; \end{cases} \quad \text{and} \quad \begin{cases} \varepsilon_2(4)=\left[\frac{\varepsilon_2(3)+c_0(4)}{\varepsilon_3(3)c_0(4)+1}\right]=1; \\ \varepsilon_3(4)\left[\frac{\varepsilon_3(3)}{c_0(4)+1}\right]=0; \\ c_1(4)=c_0(4)\varepsilon_3(3)=0. \end{cases}$$

Hence, $\varepsilon_3(4)\varepsilon_2(4)\varepsilon_1(4)\varepsilon_0(4)=0101=4=N+1$.

b) Let $N=4=00101=\varepsilon_4(4)\varepsilon_3(4)\varepsilon_2(4)\varepsilon_1(4)\varepsilon_0(4)$. By description (1.2) we have $\varepsilon_0(5)=0$, $c_0(5)=1$ and

$$\begin{cases} \varepsilon_1(5)=\left[\frac{\varepsilon_1(4)+c_0(5)}{\varepsilon_2(4)c_0(5)+1}\right]=0; \\ \varepsilon_2(5)\left[\frac{\varepsilon_2(4)}{c_0(5)+1}\right]=0; \\ c_1(5)=c_0(5)\varepsilon_2(4)=1; \end{cases} \quad \text{and} \quad \begin{cases} \varepsilon_3(5)=\left[\frac{\varepsilon_3(4)+c_1(5)}{\varepsilon_4(4)c_1(5)+1}\right]=1; \\ \varepsilon_4(5)\left[\frac{\varepsilon_4(4)}{c_1(5)+1}\right]=0; \\ c_2(5)=c_1(5)\varepsilon_4(5)=0. \end{cases}$$

Hence, $\varepsilon_4(5)\varepsilon_3(5)\varepsilon_2(5)\varepsilon_1(5)\varepsilon_0(5)=01000=5=N+1$.

Theorem 1.6. *Let N be a nonnegative integer number and $\varepsilon_k(N)\dots\varepsilon_0(N)$ its representation in Fibonacci base. The algorithms (1.1) and (1.2) give for us the digits of $N+1$ in Fibonacci base.*

Proof. Let $N=\varepsilon_k(N)\dots\varepsilon_0(N)$ and assume that $\varepsilon_0(N)=0$.

If $\varepsilon_1(N)=0$ then by de system (1.1) we have $\varepsilon_0(N+1)=1$, $\varepsilon_1(N+1)=0$, $c_0(N+1)=0$ and, by the remark 1.4, $\varepsilon_i(N+1)=\varepsilon_i(N)$, for all $i\geqslant 2$. Therefore, $\varepsilon_k(N+1)\dots\varepsilon_2(N+1)\varepsilon_1(N+1)\varepsilon_0(N+1)=\varepsilon_k(N)\dots\varepsilon_2(N)01=N+1$.

If $N = \varepsilon_k(N)\dots\varepsilon_{2l+2}(N)00(10)^l$, for some $l \geqslant 1$ then $\varepsilon_{2i+1}(N)\varepsilon_{2i}(N) = 10$, for all $i \in \{0,\dots,l-1\}$, and $\varepsilon_{2l+1}(N)\varepsilon_{2l}(N) = 00$. Thus, by the system (1.1), we have $\varepsilon_{2i+1}(N)\varepsilon_{2i}(N) = 00$ and $c_i(N+1) = 1$, for all $i \in \{0,\dots,l-1\}$, $\varepsilon_{2l+1}(N)\varepsilon_{2l}(N) = 01$ and $c_l(N+1) = 0$. Therefore, by remark 1.4, $\varepsilon_i(N+1) = \varepsilon_i(N)$, for all $i \geqslant 2l+2$, and $\varepsilon_k(N+1)\dots\varepsilon_{2l+2}(N+1)\varepsilon_{2l+1}(N+1)\varepsilon_{2l}(N+1)\varepsilon_{2l-1}(N+1)\dots\varepsilon_0(N+1) =$
$\varepsilon_k(N)\dots\varepsilon_{2l+2}(N)01\underbrace{0\dots0}_{2l-1} = N+1$.

The case $\varepsilon_0(N) = 1$ can be done by the same way using the system (1.2). $\square$

1.2. **Stochastic adding machine.** A way to construct the stochastic Fibonacci adding machine is by a "probabilistic" transducer $\mathcal{T}_{\overline{p}}$ (see Fig.2).

The states of $\mathcal{T}_{\overline{p}}$ are T and I_i, for all $i \geqslant 1$. The labels are of the form $(0/0,1)$, $(1/1,1)$, $(a/b,p_i)$ or $(a/a,1-p_i)$, for all $i \geqslant 1$, where a/b is a label in $\mathcal{T}$. The labelled edges in $\mathcal{T}_{\overline{p}}$ are of the form $(T,(x/x,1),T)$ where $x \in \{0,1\}$, $(I_{i+1},(a/b,p_i),I_i)$ and $(T,(a/a,1-p_i),I_i)$ where $(I,a/b,I)$ is a labelled edge in $\mathcal{T}$ or $(T,(a/b,p_i),I_i)$ and $(T,(a/a,1-p_i),I_i)$ where $(T,a/b,I)$ is a labelled edge in $\mathcal{T}$, for all $i \geqslant 1$.

The stochastic process $\psi(N)$, with state space $\mathbb{N}$, is defined by $\psi(N) = \sum_{i=0}^{+\infty}\varepsilon_i(N)F_i$ where $(\varepsilon_i(N))_{i\geqslant 0}$ is an infinite sequence of 0 or 1 without two 1 consecutive and with finitely many no zero terms. The sequence $(\varepsilon_i(N))_{i\geqslant 0}$ is defined in the following way:

Put $\varepsilon_i(0) = 0$ for all i and assume that we have defined $(\varepsilon_i(N-1))_{i\geqslant 0}$ with $N \geqslant 1$. In the transducer $\mathcal{T}_{\overline{p}}$, consider the path

$$\dots(T,(0/0,1),T)\dots(T,(0/0,1),T)(s_{n+1},(a_n/b_n,t_n),s_n)\dots(s_1,(a_0/b_0,t_0),s_0)$$

where $s_0 = I_1$ and $s_{n+1} = T$, such the words $\dots00\varepsilon_n(N-1)\dots\varepsilon_0(N-1)$ and $\dots00a_n\dots a_0$ are equal. We define the sequence $(\varepsilon_i(N))_{i\geqslant 0}$ as the infinite sequence whose terms are 0 or 1 such that $\dots00\varepsilon_n(N)\dots\varepsilon_0(N) = \dots00b_n\dots b_0$.

We remark that $\psi(N-1)$ transitions to $\psi(N)$ with probability $p_{\psi(N-1)\psi(N)} = t_nt_{n-1}\dots t_0$.

Example 1.7. If $N = 17 = 100101$ then in the transducer $\mathcal{T}$ of Fibonacci adding machine, N corresponds to the path $(T,1/1,T)(T,00/01,I)(I,10/00,I)(I,1/0,I)$. In the transducer $\mathcal{T}_{\overline{p}}$ of the stochastic Fibonacci adding machine we have the followings paths:
a) $(T,(1/1,1),T)(T,(0/0,1),T)(T,(0/0,1),T)(T,(1/1,1),T)(T,(0/0,1),T)(T,(1/1,1-p_1),I_1)$. In this case $N = 17$ transitions to 17 with probability $1-p_1$.
b) $(T,(1/1,1),T)(T,(0/0,1),T)(T,(0/0,1),T)(T,(10/10,1-p_2),I_2)(I_2,(1/0,p_1),I_1)$. In this case $N = 17$ transitions to 16 with probability $p_1(1-p_2)$.
c) $(T,(1/1,1),T)(T,(00/00,1-p_3),I_3)(I_3,(10/00,p_2),I_2)(I_2,(1/0,p_1),I_1)$. In this case $N = 17$ transitions to 13 with probability $p_1p_2(1-p_3)$.
d) $(T,(1/1,1),T)(T,(00/01,p_3),I_3)(I_3,(10/00,p_2),I_2)(I_2,(1/0,p_1),I_1)$. In this case $N = 17$ transitions to 18 with probability $p_1p_2p_3$.

An other way to construct the stochastic Fibonacci adding machine is the same that Killeen and Taylor did on description (0.2) generalizing the descriptions (1.1) and (1.2) to include fallible adding machines by the following manner: let $\overline{p} = (p_i)_{i\geqslant 1} \subset]0,1]$ be a probabilities sequence and $\{e_i(N) : i \geqslant 0\ \ N \in \mathbb{N}\}$ be an independent, identically distributed family of random variables parametrized by natural numbers i,n where, for each nonnegative integer N, we have $e_i(N) = 0$ with probability $1-p_{i+1}$ and $e_i(N) = 1$ with probability p_{i+1}, for all $i \geqslant 0$.

Let N a nonnegative integer. Given a sequence $(\varepsilon_i(N))_{i\geqslant 0}$ of $0's$ and $1's$ such that $\varepsilon_i(N)=1$ for finitely many indices i and $\varepsilon_{i+1}(N)\varepsilon_i(N)\neq 11$, for all $i\geqslant 0$, we consider the sequences $(\varepsilon_i(N+1))_{i\geqslant 0}$ and $(c_i(N+1))_{i\geqslant -1}$, defined by $c_{-1}(N+1)=1$ and

$a)$ If $\varepsilon_0(N)=0$ then, for all $i\geqslant 0$, do

$$\begin{cases} \varepsilon_{2i}(N+1)=\left[\frac{\varepsilon_{2i}(N)+e_i(N)c_{i-1}(N+1)}{e_i(N)\varepsilon_{2i+1}(N)c_{i-1}(N+1)+1}\right]; \\ \varepsilon_{2i+1}(N+1)=\left[\frac{\varepsilon_{2i+1}(N)}{e_i(N)c_{i-1}(N+1)+1}\right]; \\ c_i(N+1)=e_i(N)c_{i-1}(N+1)\varepsilon_{2i+1}(N). \end{cases}$$

$b)$ If $\varepsilon_0(N)=1$ then $\varepsilon_0(N+1)=\left[\frac{1}{e_0(N)+1}\right]$, $c_0(N+1)=e_0(N)$ and, for all $i\geqslant 1$, do

$$\begin{cases} \varepsilon_{2i-1}(N+1)=\left[\frac{\varepsilon_{2i-1}(N)+e_i(N)c_{i-1}(N+1)}{e_i(N)\varepsilon_{2i}(N)c_{i-1}(N+1)+1}\right]; \\ \varepsilon_{2i}(N+1)=\left[\frac{\varepsilon_{2i}(N)}{e_i(N)c_{i-1}(N+1)+1}\right]; \\ c_i(N+1)=e_i(N)c_{i-1}(N+1)\varepsilon_{2i}(N). \end{cases}$$

The systems $a)$ and $b)$ gives to us the Fibonacci stochastic adding machine.

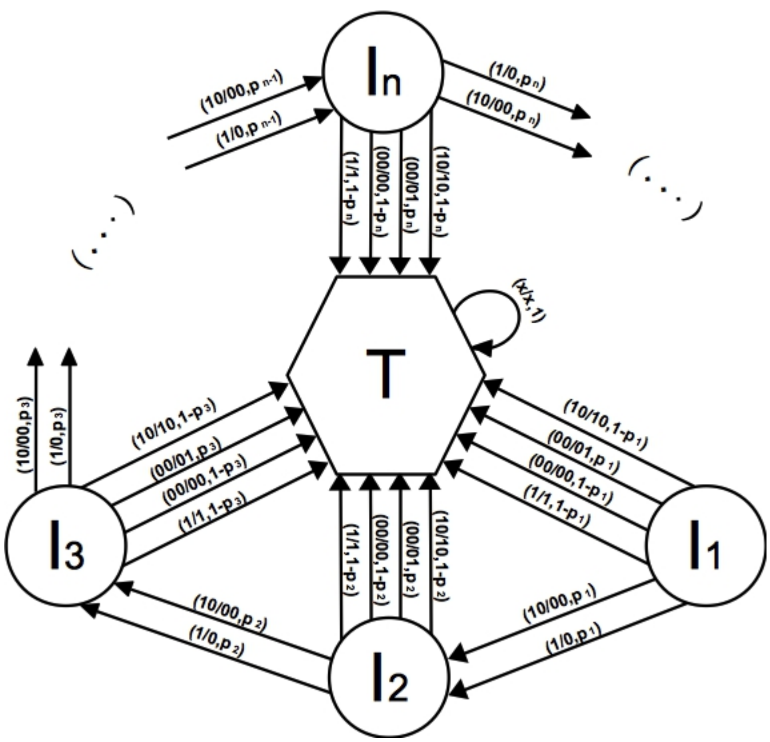


FIGURE 2. Transducer $\mathcal{T}_{\overline{p}}$ of the stochastic Fibonacci adding machine.

With the transition probabilities, we obtain the countable transition matrix of the Markov chain $S=S_{\overline{p}}=(S_{i,j})_{i,j\geqslant 0}$. This Markov chain is irreducible and aperiodic. To help the reader, the first entries of the matrix S are given by table 1.

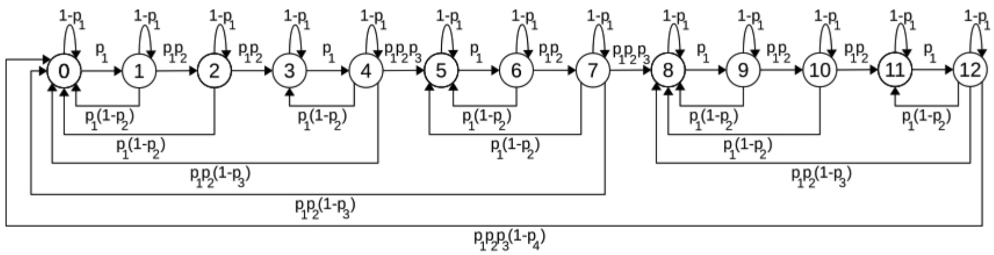


FIGURE 3. Transition graph of the stochastic Fibonacci adding machine.

$$\begin{pmatrix}
1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
p_1(1-p_2) & 1-p_1 & p_1p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
p_1(1-p_2) & 0 & 1-p_1 & p_1p_2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & \dots \\
p_1p_2(1-p_3) & 0 & 0 & p_1(1-p_2) & 1-p_1 & p_1p_2p_3 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & p_1(1-p_2) & 1-p_1 & p_1p_2 & 0 & 0 & \dots \\
p_1p_2(1-p_3) & 0 & 0 & 0 & 0 & p_1(1-p_2) & 0 & 1-p_1 & p_1p_2p_3 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_1 & p_1 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1(1-p_2) & 1-p_1 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1(1-p_2) & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
p_1p_2p_3(1-p_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1p_2(1-p_3) & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

TABLE 1. First entries of the transition operator S.

Proposition 1.8. *Let N be a nonnegative integer. Then the following results are satisfied.*
i) *N transitions to N with probability $1-p_1$.*
ii) *If $N = \varepsilon_k \dots \varepsilon_0$ with $\varepsilon_1\varepsilon_0 = 00$ then N transitions to $N+1$ with probability p_1.*
iii) *If $N = \varepsilon_k \dots \varepsilon_t(10)^s$ with $\varepsilon_{t+1}\varepsilon_t = 00$, $s \geqslant 1$ and $t = 2s$ then N transitions to $N+1$ with probability $p_1p_2 \dots p_{s+1}$ and N transitions to $N - F_{2m} + 1$, $0 < m \leqslant s$, with probability $p_1p_2 \dots p_m(1-p_{m+1})$.*
iv) *If $N = \varepsilon_k \dots \varepsilon_t(10)^s 1$ with $\varepsilon_{t+1}\varepsilon_t = 00$, $s \geqslant 1$ and $t = 2s+1$ then N transitions to $N+1$ with probability $p_1p_2 \dots p_{s+2}$ and N transitions to $N - F_{2m-1} + 1$, $0 < m \leqslant s+1$, with probability $p_1p_2 \dots p_m(1-p_{m+1})$.*

Proof. **(ii)** If $N = \varepsilon_k \dots \varepsilon_0$ with $\varepsilon_1\varepsilon_0 = 00$ then $(T, (\varepsilon_k/\varepsilon_k, 1), T) \dots (T, (\varepsilon_2/\varepsilon_2, 1), T)(T, (00/01, p_1), I_1)$ is a path in the transducer $\mathcal{T}_{\overline{p}}$. Then N transitions to $N+1$ with probability p_1.
(iii) If $N = \varepsilon_k \dots \varepsilon_t(10)^s$ then $\varepsilon_{t+1}\varepsilon_t = 00$, $s \geqslant 1$ and $t = 2s$. Therefore,

a) $(T, (\varepsilon_k/\varepsilon_k, 1), T) \dots (T, (\varepsilon_{t+2}/\varepsilon_{t+2}, 1), T)(T, (00/01, p_{s+1}), I_{s+1})$ $\underbrace{(I_{s+1}, (10/00, p_s), I_s) \dots (I_3, (10/00, p_2), I_2)(I_2, (10/00, p_1), I_1)}_{s}$ is a path in the transducer $\mathcal{T}_{\overline{p}}$. In this case, N transitions to $N+1$ with probability $p_1 \dots p_s p_{s+1}$.

b) $(T, (\varepsilon_k/\varepsilon_k, 1), T) \dots (T, (\varepsilon_t/\varepsilon_t, 1), T)$ $\underbrace{(T, (1/1, 1), T)(T, (0/0, 1), T) \dots (T, (1/1, 1), T)(T, (0/0, 1), T)}_{2s-2m-2}(T, (10/10, 1-p_{m+1}), I_{m+1})$ $(I_{m+1}, (10/00, p_m), I_m) \dots (I_3, (10/00, p_2), I_2)(I_2, (10/00, p_1), I_1)$ is a path in the transducer $\mathcal{T}_{\overline{p}}$. In this case, N transitions to $N - F_{2m} + 1$ with probability $p_1 \dots p_m(1-p_{m+1})$.

The cases (i) and (iv) are left to the reader. $\square$

Proposition 1.9. *Let $i, j \in \mathbb{N}^*$ such that $F_n \leqslant i, j \leqslant F_{n+1} - 1$. Then the transition probability from i to j is equal to transition probability from $i - F_n$ to $j - F_n$, i.e. $S_{i,j} = S_{i-F_n, j-F_n}$.*

Proof. If $i = j$ then by proposition 1.8 we have that $S_{i,j} = S_{i-F_n,j-F_n} = 1 - p_1$.

Suppose $i \neq j$ and $S_{i,j} > 0$.

$i)$ If $i = \varepsilon_n \dots \varepsilon_2 00$ with $\varepsilon_n = 1$ then by the item (ii) of the proposition 1.8, $j = i + 1 = \varepsilon_n \dots \varepsilon_2 01$ and $S_{i,j} = p_1$. Thus $i - F_n = \varepsilon_{n-1} \dots \varepsilon_2 00$, $j - F_n = \varepsilon_{k-1} \dots \varepsilon_2 01$ and $S_{i-F_n,j-F_n} = p_1 = S_{i,j}$.

$ii)$ If $i = \varepsilon_n \dots \varepsilon_t 00 \underbrace{1010 \dots 1010}_{2s}$, with $s \geqslant 1$ then by the item (iii) of the proposition 1.8, we have two cases: $j = \varepsilon_n \dots \varepsilon_t 01 \underbrace{0000 \dots 0000}_{2s}$ and $S_{i,j} = p_1 p_2 \dots p_{s+1}$ or $j = i - F_{2m} + 1 = \varepsilon_n \dots \varepsilon_t 00 \underbrace{10 \dots 10}_{2s-2m} \underbrace{00 \dots 00}_{2m}$, with $1 \leqslant m \leqslant s$ e $S_{i,j} = p_1 p_2 \dots p_m (1 - p_{m+1})$. Since $F_n \leqslant i, j \leqslant F_{n+1} - 1$ then the Fibonacci expansions of i and j are $i = 1c_{n-1} \dots c_0$ and $j = 1d_{n-1} \dots d_0$, where $c_l, d_l \in \{0, 1\}$, $c_{n-1} = d_{n-1} = 0$, $c_{l+1} c_l \neq 11$ and $d_{l+1} d_l \neq 11$, for all $l \in \{0, \dots, n-2\}$. Therefore, in the first case there exists a integer $t \leqslant n_0 \leqslant n - 2$ such that $\varepsilon_{n_0} = 1$ and $\varepsilon_l = 0$ for all integer $n_0 < l \leqslant n - 1$. Hence, we have $i - F_n = \varepsilon_{n_0} \dots \varepsilon_t 00 \underbrace{1010 \dots 1010}_{2s}$, $j - F_n = \varepsilon_{n_0} \dots \varepsilon_t 01 \underbrace{0000 \dots 0000}_{2s}$ and $S_{i-F_n,j-F_n} = p_1 p_2 \dots p_{s+1} = S_{i,j}$.

In the second case there exists a integer n_0 such that $n_0 = 2s - 1$ or $t \leqslant n_0 \leqslant n - 2$ such that $\varepsilon_{n_0} = 1$ and $\varepsilon_l = 0$ for all $n_0 < l \leqslant n - 1$. Hence, we have $i - F_n = \varepsilon_{n_0} \dots \varepsilon_t 00 \underbrace{1010 \dots 1010}_{2s}$ and $j - F_n = \varepsilon_{n_0} \dots \varepsilon_t 00 \underbrace{10 \dots 10}_{2s-2m} \underbrace{00 \dots 00}_{2m}$ (case $t \leqslant n_0 \leqslant n - 2$) or $i - F_n = 00 \underbrace{1010 \dots 1010}_{2s-2}$ and $j - F_n = 00 \underbrace{10 \dots 10}_{2s-2-2m} \underbrace{00 \dots 00}_{2m}$ (case $n_0 = 2s - 1$). Therefore, $S_{i-F_n,j-F_n} = p_1 p_2 \dots p_m (1 - p_{m+1}) = S_{i,j}$.

$iii)$ Using the same idea used in the proof of the anterior item, we deduce the case that $N = \varepsilon_n \dots \varepsilon_t 00 \underbrace{1010 \dots 1010}_{2s} 1$. $\square$

Proposition 1.10. *Let $i, j, n \in \mathbb{N}^*$ such that $0 < i < F_n \leqslant j$. Then $S_{j,i} = 0$.*

Proof. Let $n \in \mathbb{N}$ such that $0 < i < F_n \leqslant j < F_{n+1}$. Hence, the expressions of i and j in Fibonacci base are

$$i = \varepsilon'_{n-1} \dots \varepsilon'_0 \text{ and } j = 1\varepsilon_{n-1} \dots \varepsilon_0, \tag{1.3}$$

where $\varepsilon_l, \varepsilon'_l \in \{0, 1\}$, $\varepsilon_{l+1} \varepsilon_l \neq 11$ and $\varepsilon'_{l+1} \varepsilon'_l \neq 11$ for all $l \in \{0, \dots, n\}$. If $S_{j,i} \neq 0$ then from proposition 1.8 and the fact that $i < j$, we deduce that $j = \varepsilon_n \dots \varepsilon_t 00 \underbrace{10 \dots 10}_{2s}$, with $s \geqslant 1$ and $n \geqslant t \geqslant 2s + 2$, and $i = j - F_{2m} + 1 = \varepsilon_n \dots \varepsilon_t 00 \underbrace{10 \dots 10}_{2s-2m} \underbrace{00 \dots 00}_{2m}$,

with $1 \leqslant m \leqslant s$, or $j = \varepsilon_n \dots \varepsilon_t 00 \underbrace{10 \dots 10}_{2s-2} 1$, with $s \geqslant 2$ and $n \geqslant t \geqslant 2s+1$, and $i = j - F_{2m+1} + 1 = \varepsilon_n \dots \varepsilon_t 00 \underbrace{10 \dots 10}_{2s-2m} \underbrace{00 \dots 000}_{2m-1}$, with $1 \leqslant m \leqslant s$.

In both cases we obtain a contraction because of the relation (1.3). □

2. Probabilistic properties of the stochastic Fibonacci adding machine

In the next proposition, we obtain a necessary and sufficient condition for transience of the Markov chain.

Proposition 2.1. *The Markov chain is transient if only if* $\prod_{i=1}^{\infty} p_i > 0$.

For the proof of the proposition 2.1, we need the following definition and lemma:

Lemma 2.2. [O] *An irreducible and aperiodic Markov chain is transient if only if there exists a sequence* $v = (v_i)_{i \geqslant 1}$ *such that* $0 < v_i \leqslant 1$ *and* $v_i = \sum_{j=1}^{+\infty} S_{i,j} v_j$ *for all* $i \geqslant 1$, *i.e.* $\tilde{S}v = v$ *where* $\tilde{S}$ *is obtained from* S *by removing its first line and column.*

Indeed, in the transient case a solution is obtained by taking v_m as the probability that 0 is never visited by the Markov chain given that the chain starts at state m, see the discussion on pp. $42 - 43$, Chap. 2 in [L] and also Corollary 16.48 in [O].

Definition 2.3. Let $\beta : \mathbb{N}^* \longrightarrow \mathbb{R}_+$ be the map defined by $\beta(n) = \Pi_r$ if $F_r \leqslant n < F_{r+1}$, for all $r \geqslant 0$, where $\Pi_0 = 1$, $\Pi_1 = \frac{1}{p_2}$ and $\Pi_r := \prod_{i=2}^{[\frac{r}{2}]+1} \frac{1}{p_i^2} \frac{1}{p_{[\frac{r}{2}]+2}^{[\frac{r+1}{2}]-[\frac{r}{2}]}}$, for all $r \geqslant 2$, i.e.

$$\Pi_{2n} = \prod_{i=2}^{n+1} \frac{1}{p_i^2} = \frac{1}{p_2^2 p_3^2 \dots p_{n+1}^2} \text{ and } \Pi_{2n+1} = \frac{1}{p_{n+2}} \prod_{i=2}^{n+1} \frac{1}{p_i^2} = \frac{1}{p_2^2 p_3^2 \dots p_{n+1}^2} \frac{1}{p_{n+2}}, \text{ for all } n \geqslant 1.$$

Lemma 2.4. *Let* $v = (v_i)_{i \geqslant 1} \in l^\infty(\mathbb{N})$ *be a bounded sequence. Then* $\tilde{S}v = v$ *if only if* $v_n = \beta(n) v_1$ *for all* $n \geqslant 1$, *where* $\tilde{S}$ *is the matrix obtained from* S *by removing its first line and column.*

Proof. Assume that $\tilde{S}v = v$. Since $S_{i,i+k} = 0$ for all integer $i \geqslant 0$ and $k \geqslant 2$, (see the proposition 1.8) then the operator $\tilde{S}$ satisfies $\tilde{S}_{i,i+k} = S_{i,i+k} = 0$, for all integers $i \geqslant 1$ and $k \geqslant 2$. Therefore, it is possible to prove by induction that for each integer $n \geqslant 1$, there exists $\alpha_n \in \mathbb{R}$ such that $v_n = \alpha_n v_1$.

Claim: $\alpha_n = \beta(n)$, for all integer $n \geqslant 1$.

Indeed, since $(\tilde{S} - I)v = 0$, it's not hard to see that $\alpha_1 = 1 = \beta(1)$, $\alpha_2 = \frac{1}{p_2} = \beta(2)$ and $\alpha_3 = \alpha_4 = \frac{1}{p_2^2} = \beta(3) = \beta(4)$.

Let $k \in \mathbb{N}$, $k \geqslant 5$, and suppose that $\alpha_n = \beta(n)$, for all $n \in \{1, \dots, k\}$.

To prove $\alpha_{k+1} = \beta(k+1)$, we need to consider two cases:

(i) $k+1 = F_r$ for some $r \geqslant 4$;

In this case we can have $r = 2l+1$ or $r = 2l+2$, for some $l \geqslant 1$. In both cases we have

$$\begin{cases} S_{k,k+1} = p_1 p_2 \dots p_{l+1} p_{l+2}; \\ S_{k,0} = p_1 p_2 \dots p_{l+1}(1-p_{l+2}); \\ S_{k,\sum_{i=0}^{j} F_{r-1-2i}} = p_1 p_2 \dots p_{l-j}(1-p_{l-j+1}), \text{ for all } j = 0\dots, l-1; \\ S_{k,k} = 1-p_1. \end{cases}$$

Since $(\tilde{S} - Iv)_k = 0$, we have

$$p_1 p_2 \dots p_{l+1} p_{l+2} v_{k+1} + \sum_{j=0}^{l-1} p_1 p_2 \dots p_{l-j}(1-p_{l-j+1}) v_{\sum_{i=0}^{j} F_{r-1-2i}} + (1-p_1) v_k = v_k. \tag{2.1}$$

Thus, since $F_{r-1} \leqslant \sum_{i=0}^{j} F_{r-1-2i} \leqslant \sum_{i=0}^{l-1} F_{r-1-2i} < F_r = k+1$, for all $j \in \{0, \dots, l-1\}$ and $\alpha_n = \beta(n)$, for all $n \in \{1, \dots, k\}$ then $v_{\sum_{i=0}^{j} F_{r-1-2i}} = v_k = \Pi_{r-1} v_1$, for all $j = 0, \dots, l-1$. Therefore, from relation (2.1), we obtained

$$p_1 p_2 \dots p_{l+1} p_{l+2} v_{k+1} + \sum_{j=0}^{l-1} p_1 p_2 \dots p_{l-j}(1-p_{l-j+1}) \Pi_{r-1} v_1 + (1-p_1) \Pi_{r-1} v_1 = \Pi_{r-1} v_1.$$

Then it implies that $p_1 p_2 \dots p_{l+1} p_{l+2} v_{k+1} = p_1 p_2 \dots p_{l+1} \Pi_{r-1} v_1$ i.e.

$$v_{k+1} = \frac{1}{p_{l+2}} \Pi_{r-1} v_1 = \frac{1}{p_{l+2}} \prod_{i=2}^{[\frac{r-1}{2}]+1} \frac{1}{p_i^2} \frac{1}{p_{[\frac{r-1}{2}]+2}^{[\frac{r-1+1}{2}]-[\frac{r-1}{2}]}} v_1.$$

If $r = 2l+1$ then $v_{k+1} = \prod_{i=2}^{l+1} \frac{1}{p_i^2} \frac{1}{p_{l+2}} v_1 = \Pi_r v_1$ i.e. $\alpha_{k+1} = \beta(k+1)$.

If $r = 2l+2$ then $v_{k+1} = \prod_{i=2}^{l+2} \frac{1}{p_i^2} v_1 = \Pi_r v_1$ i.e. $\alpha_{k+1} = \beta(k+1)$.

(ii) $F_r < k+1 < F_{r+1}$, for some $r \geqslant 3$;
In this case, we have that $F_r \leqslant k < F_{r+1} - 1$. Then by proposition 1.10 it implies that $S_{k,i} = 0$, for all $i \in \{0, \dots, F_r - 1\}$. Thus, we have that

$$1 = \textstyle\sum_{i=0}^{k+1} S_{k,i} = \sum_{i=F_r}^{k+1} S_{k,i} = \sum_{i=F_r}^{k+1} \tilde{S}_{k,i} \text{ and } 1 - S_{k,k+1} = \sum_{i=F_r}^{k} \tilde{S}_{k,i}.$$

Therefore, since $(\tilde{S} - I)v = 0$ and $\alpha_n = \beta(n) = \Pi_r$, for all $n \in \{F_r, \dots, k\}$, we have

$$\Pi_r v_1 = v_k = \textstyle\sum_{i=1}^{k+1} \tilde{S}_{k,i} v_i = \sum_{i=F_r}^{k} \tilde{S}_{k,i} \Pi_r v_1 + \tilde{S}_{k,k+1} v_{k+1} = (1 - \tilde{S}_{k,k+1}) \Pi_r v_1 + \tilde{S}_{k,k+1} v_{k+1}.$$

Thus, it follows that $\tilde{S}_{k,k+1} v_{k+1} = \tilde{S}_{k,k+1} \Pi_r v_1$, i.e. $\alpha_{k+1} = \Pi_r = \beta(k+1)$. □

Proof of the proposition 2.1: Suppose that the Markov chain is transient. Then there exists $v = (v_i)_{i \geqslant 1}$ satisfying the conditions of the lemma 2.2. By lemma 2.4, we have that $v_n = \beta(n) v_1$, for all $n \geqslant 1$. Thus, we need to have $\prod_{i=1}^{+\infty} p_i > 0$, because otherwise we should have $v_n = \beta(n) v_1$ goes to infinity, contradicting the fact that $|v_i| \leqslant 1$ for all $i \geqslant 1$. □

Proposition 2.5. *Let $\mu = (\mu_i)_{i\geqslant 0}$ be a sequence on $\mathbb{C}$. Then $\mu S = \mu$ if only if there exists a sequence $(\xi_i)_{i\geqslant 1} = (\xi_i(\overline{p}))_{i\geqslant 1}$ such that $\mu_i = \xi_i\mu_0$, for all integer $i \geqslant 1$, where $\xi_1 = 1$, $\xi_2 = p_2$ and $\xi_{F_n+k} = \xi_{F_n}\xi_k$, for all integer $n \geqslant 2$ and $k \in \{1, \dots, F_{n-1} - 1\}$. Furthermore, $\xi_{F_n} = p_{\left[\frac{n+1}{2}\right]+1}$ and $\xi_{F_n-1} = p_2p_3 \dots p_{\left[\frac{n}{2}\right]+1}$.*

Proof. Assume that $\mu S = \mu$. Since for each nonnegative integer $n \geqslant 1$ we have $S_{i,j} = 0$, with $0 < j < F_n \leqslant i$ (see proposition 1.10) then $\mu_j = \sum_{i=0}^{+\infty} \mu_i S_{i,j} = \sum_{i=0}^{F_n-1} \mu_i S_{i,j}$, for all integer $n \geqslant 1$ and $j \in \{1, \dots, F_n - 1\}$. Then it's possible to prove by induction that there exists $(\xi_j)_{j\geqslant 1} = (\xi_j(\overline{p}))_{j\geqslant 1} \in l^\infty(\mathbb{N})$ such that $\mu_j = \xi_j\mu_0$, for all $j \geqslant 1$. Furthermore, it's not hard to prove that $\xi_1 = 1$, $\xi_2 = p_2$, $\xi_{F_2} = \xi_3 = p_2 = p_{\left[\frac{2+1}{2}\right]+1}$ and $\xi_{F_2+1} = \xi_{F_3-1} = \xi_4 = p_2 = \xi_1\xi_{F_2}$.

Let $n \in \mathbb{N}$, $n \geqslant 3$ and $k \in \{1, \dots, F_{n-1} - 1\}$. Since $S_{i,F_n+k} = 0$, for each $i \in \{0, \dots, F_n + k - 2\}$, we have that $\mu_{F_n+k} = \sum_{i=0}^{+\infty} \mu_i S_{i,F_n+k} = \sum_{i=F_n+k-1}^{+\infty} \mu_i S_{i,F_n+k}$.

Then from proposition 1.9, we have that

$$\mu_{F_n+k} = \sum_{i=F_n+k-1}^{+\infty} \mu_i S_{i,F_n+k} = \sum_{i=F_n+k-1}^{+\infty} \mu_i S_{i-F_n,k} = \sum_{i=k-1}^{+\infty} \mu_{i+F_n} S_{i,k} = \sum_{i=0}^{+\infty} \mu_{i+F_n} S_{i,k}.$$

Let $(\mu'_i)_{i\geqslant 0}$ defined by $\mu'_i = \mu_{F_n+i}$. Hence $\mu'_k = \sum_{i=0}^{+\infty} \mu'_i S_{i,k} = \sum_{i=0}^{F_n-1} \mu'_i S_{i,k}$, for all $k \in \{1, \dots, F_{n-1} - 1\}$.

Therefore, $\mu'_k = \xi_k\mu'_0$, i.e. $\mu_{F_n+k} = \xi_k\mu_{F_n}$. Since $\mu_{F_n} = \xi_{F_n}\mu_0$ and $\mu_{F_n+k} = \xi_{F_n+k}\mu_0$, it follows that $\xi_{F_n+k} = \xi_{F_n}\xi_k$, for all $k \in \{1, \dots, F_{n-1} - 1\}$.

Now we have that $\xi_{F_n} = p_{\left[\frac{n+1}{2}\right]+1}$ and $\xi_{F_m-1} = p_2p_3 \dots p_{\left[\frac{m}{2}\right]+1}$ is true for $n = 1, 2$ and $m = 2, 3$. Let $k > 2$ a integer number and supposes that $\xi_{F_n} = p_{\left[\frac{n+1}{2}\right]+1}$ and $\xi_{F_m-1} = p_2p_3 \dots p_{\left[\frac{m}{2}\right]+1}$, for all $n \in \{1, \dots, k-1\}$ and $m \in \{2, \dots, k\}$. Assume $\mu = \mu S$. Then

$$\mu_{F_k} = \sum_{i=0}^{+\infty} \mu_i S_{i,F_k} = \sum_{i=F_k-1}^{+\infty} \mu_i S_{i,F_k} = \mu_{F_k-1}S_{F_k-1,F_k} + \mu_{F_k}S_{F_k,F_k} + \sum_{i=F_k+1}^{F_k+F_{k-1}-1} \mu_i S_{i,F_k},$$

On the other hand, by the first part of proposition, we have that

$$\sum_{i=F_k+1}^{F_k+F_{k-1}-1} \mu_i S_{i,F_k} = \sum_{i=F_k+1}^{F_k+F_{k-1}-1} \mu_i S_{i-F_k,0} = \sum_{i=1}^{F_{k-1}-1} \mu_{F_k+i} S_{i,0} = \left(\sum_{i=1}^{F_{k-1}-1} \xi_i S_{i,0}\right) \mu_{F_k} = \left(\sum_{i=1}^{n-1} \xi_{F_i-1} S_{F_i-1,0}\right) \mu_{F_k}.$$

Thus, from proposition 1.8, it follows that

$$\mu_{F_k} = p_2p_3 \dots p_{\left[\frac{k}{2}\right]+1}\mu_0 p_1p_2 \dots p_{\left[\frac{k+1}{2}\right]+1} + (1-p_1)\mu_{F_k} +$$
$$+ (p_1(1-p_2) + p_2p_1(1-p_2) + p_2p_1p_2(1-p_3) + p_2p_3p_1p_2(1-p_3) + p_2p_3p_1p_2p_3(1-p_4) +$$
$$+ \dots + p_2p_3 \dots p_{\left[\frac{k-1}{2}\right]+1} p_1p_2 \dots p_{\left[\frac{k}{2}\right]} \left(1 - p_{\left[\frac{k}{2}\right]+1}\right)\Big) \mu_{F_k}.$$

Hence, it follows that $p_2p_3 \dots p_{\left[\frac{k-1}{2}\right]+1}\mu_{F_k} = p_2p_3 \dots p_{\left[\frac{k+1}{2}\right]+1}\mu_0$, i.e. $\mu_{F_k} = p_{\left[\frac{k+1}{2}\right]+1}\mu_0$.

On the other hand, we have that

$$\xi_{F_{k+1}-1} = \xi_{F_k+F_{k-1}-1} = \xi_{F_{k-1}-1}\xi_{F_k} = p_2p_3\dots p_{\left[\frac{k-1}{2}\right]+1}p_{\left[\frac{k+1}{2}\right]+1} = p_2p_3\dots p_{\left[\frac{k+1}{2}\right]+1}.$$

□

Theorem 2.6. *If $\prod_{i=1}^{+\infty} p_i = 0$ and $\sum_{i=1}^{+\infty} p_i = +\infty$ then the Markov chains is null recurrent.*

Proof. Since $\prod_{i=1}^{+\infty} p_i = 0$ then by the theorem 2.1 we have that the Markov chain is recurrent.

Let $\mu = (\mu_0, \mu_1, \mu_2, \dots) \in l^1(\mathbb{N})$ be a invariant measure for the operator S. Then $\mu S = \mu$. Hence, from proposition 2.5, it follows that $\|\mu\|_1 = \sum_{i=0}^{+\infty} |\mu_i| \geqslant \sum_{i=1}^{+\infty} |\mu_{F_i}| = \sum_{i=2}^{+\infty} 2p_i = +\infty$, which yields a contradiction to the fact that $\mu \in l^1(\mathbb{N})$. Hence, if $\sum_{i=1}^{+\infty} p_i = +\infty$ then S has no invariant probability measure and so cannot be positive recurrent.

Therefore, the Markov chain is null recurrent. □

Theorem 2.7. *There exists a sequence of probabilities $\overline{p} = (p_i)_{i\geqslant 1}$ such that satisfies the Markov chain to be positive recurrent. In particular, if $\sum_{i=2}^{+\infty} p_i F_{2(i-1)} < +\infty$ then the Markov chain is positive recurrent.*

Proof. We will construct a probabilities sequence $(p_i)_{i\geqslant 1}$, like the following:

Let $p_1, p_2 \in]0, 1]$ and $(b_n)_{n\geqslant 0} \in l^1(\mathbb{N})$, with $b_1 = 1$ and $b_2 = 3p_2$.

Let $(a_n)_{n\geqslant 0}$ be a sequence of positive real number, where $a_0 = 1 = b_1$, $a_1 = p_2$, $a_2 = 2p_2$ and for each $k \geqslant 3$, let $0 < p_k < 1$ such that $p_k \left(2 \sum_{i=0}^{2(k-2)-3} a_i + 2 + a_{2(k-2)-2}\right) \leqslant b_k$ and define $a_{2(k-1)-1} := p_k \sum_{i=0}^{2(k-2)-3} a_i + p_k$ and $a_{2(k-1)} := p_k \sum_{i=0}^{2(k-2)-2} a_i + p_k$.

Construct the Fibonacci stochastic adding machine related to the probability sequence $(p_i)_{i\geqslant 1}$ and let S be the transition operator and $\mu \in l^\infty(\mathbb{N})$ such that $\mu S = \mu$. Thus, from proposition 2.5, $\mu_i = \xi_i\mu_0$, for all $i \geqslant 0$, where $\xi_0 = 1$, $\xi_1 = 1$, $\xi_2 = p_2$ and for each $n \geqslant 2$ we have $\xi_{F_n+k} = \xi_{F_n}\xi_k$, for each $k \in \{1, \dots, F_{n-1} - 1\}$, and $\xi_{F_n} = p_{\left[\frac{n+1}{2}\right]+1}$.

Define the sequence $(\alpha_n)_{n\geqslant 0}$ by $\alpha_n = \sum_{i=F_n}^{F_{n+1}-1} \xi_i$ and note that, from proposition 2.5, for each $n \geqslant 3$, we have

$$\alpha_n = \sum_{i=F_n}^{F_{n+1}-1} \xi_i = \sum_{i=0}^{F_{n-1}-1} \xi_{F_n+i} = \xi_{F_n} \left(\sum_{i=1}^{F_{n-1}-1} \xi_i + \xi_0\right) = \xi_{F_n} \left(\sum_{i=0}^{n-2} \alpha_i + 1\right).$$

It's possible to prove by induction that $\alpha_n = a_n$, for all $n \geqslant 0$. Hence,

$$\|\mu\|_1 = |\mu_0| \sum_{i=0}^{+\infty} \xi_i = |\mu_0|\xi_0 + |\mu_0| \sum_{i=0}^{+\infty} a_i = |\mu_0| +$$

$$|\mu_0| \left(a_0 + a_1 + a_2 + \sum_{k=3}^{+\infty} (a_{2(k-1)-1} + a_{2(k-1)})\right) \leqslant |\mu_0| + |\mu_0| \left(b_1 + b_2 + \sum_{k=3}^{+\infty} b_k\right) < +\infty.$$

Therefore, $\mu \in l^1(\mathbb{N})$ and μ is a invariant measure for the operator S, i.e. the Markov chain is positive recurrent. On the other hand, we have

$$\|\mu\|_1 = |\mu_0| + |\mu_0| \sum_{i=0}^{+\infty} \alpha_i = 2|\mu_0| + |\mu_0| \sum_{k=2}^{+\infty} (\alpha_{2(k-1)-1} + \alpha_{2(k-1)}) \tag{2.2}$$

and $\alpha_n = \xi_{F_n} \sum_{i=0}^{F_{n-1}-1} \xi_i$ for all $n \geqslant 1$. From proposition 2.5, we have that $0 < \xi_i \leqslant 1$, for all $i \geqslant 0$. Wherefore, $\alpha_n \leqslant \xi_{F_n} F_{n-1}$, for all $n \geqslant 1$. Hence, we have from relation (2.2) that $\|\mu\|_1 \leqslant 2|\mu_0| + |\mu_0| \sum_{k=2}^{+\infty} p_k (F_{2(k-1)-2} + F_{2(k-1)-1}) = 2|\mu_0| + |\mu_0| \sum_{k=2}^{+\infty} p_k F_{2(k-1)}$.

Therefore, if $\sum_{k=2}^{+\infty} p_k F_{2(k-1)} < +\infty$ then $\mu \in l^1(\mathbb{N})$ and the Markov chain is positive recurrent. □

3. Spectral properties of the transition operator

Definition 3.1. Let X be a Banach space over $\mathbb{C}$ and $T : X \longrightarrow X$ be a linear operator. Then we have the following definitions:
a) *Spectrum of T:* $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bijective}\}$.
b) *Point spectrum of T:* $\sigma_{pt}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\}$.
c) *Approximate point spectrum of T:* $\sigma_a(T) = \{\lambda \in \mathbb{C} : \text{there exists a sequence } (x_n)_{n\geqslant 0} \text{ in } X, \text{ with } \|x_n\| = 1 \text{ for all } n \geqslant 0 \text{ and } \lim_{n\to\infty} \|(\lambda I - T)x_n\| = 0\}$.
d) *Continuous spectrum of T:* $\sigma_c(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is one-to-one}, \overline{(\lambda I - T)X} = X \text{ and } (\lambda I - T)X \neq X\}$.
e) *Residual spectrum of T:* $\sigma_r(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is one-to-one and } \overline{(\lambda I - T)X} \neq X\}$.

Hence, we have $\sigma(T) = \sigma_{pt}(T) \cup \sigma_c(T) \cup \sigma_r(T)$ and $\sigma_a(T) \subset \sigma(T)$.

Definition 3.2. For each $\lambda \in \mathbb{C}$, let $(q_n(\lambda))_{n\geqslant 1} = (q_n)_{n\geqslant 1}$ be a sequence defined by $q_n = q_{F_0}^{\varepsilon_0} \dots q_{F_N}^{\varepsilon_N}$, where $n = \sum_{i=0}^{N} \varepsilon_i F_i$ with $\varepsilon_i \in \{0,1\}$, $\varepsilon_{i+1}\varepsilon_i \neq 11$, $q_{F_0} = \frac{\lambda-(1-p_1)}{p_1}$, $q_{F_1} = \frac{1}{p_2} q_{F_0}^2 - \left(\frac{1}{p_2} - 1\right)$ and $q_{F_n} = \frac{1}{r_n} q_{F_{n-1}} q_{F_{n-2}} - \left(\frac{1}{r_n} - 1\right)$, where $r_n = p_{[\frac{n+1}{2}]+1}$, for all $n \geqslant 1$.

Theorem 3.3. *Acting in $l^\infty(\mathbb{N})$, we have $\sigma_{pt}(S) = \{\lambda \in \mathbb{C} : (q_n(\lambda))_{n\geqslant 0}$ is bounded$\}$.*

Remark 3.4. In particular, $\sigma_{pt}(S) \subset E := \{\lambda \in \mathbb{C} : (q_{F_n}(\lambda))_{n\geqslant 0} \text{ is bounded}\}$ and $E = \{z \in \mathbb{C} : (\psi_n(z,z))_{n\geqslant 0} \text{ is bounded}\}$, where $\psi_n = g_n \circ g_{n-1} \circ \dots \circ g_0$ and $g_n : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are maps defined by $g_0(x,y) = \left(\frac{x-(1-p_1)}{p_1}, \frac{y-(1-p_1)}{p_1}\right)$ and $g_n(x,y) = \left(\frac{1}{r_n} xy - \left(\frac{1}{r_n} - 1\right), x\right)$ for all $n \geqslant 1$.

Remark The proof of the theorem 3.3 is similar to the demonstration of the same theorem proved in [MS], where $p_i = p \in]0,1]$ is fixed, for all $i \geqslant 1$. To help the reader, the proofs will be displayed.

Before to proof the theorem 3.3, it will be necessary the followings results:

Lemma 3.5. *Let $\lambda \in \mathbb{C}$ and $v \in l^\infty(\mathbb{N})$ such that $Sv = \lambda v$. Then for each $k \geqslant 1$, there exists a complex number $\beta_k = \beta_k(\overline{p}, \lambda)$, where $\overline{p} = (p_i)_{i\geqslant 1}$, such that $v_k = \beta_k v_0$. Furthermore, if $m, n \in \mathbb{N}$, with $n > 1$, satisfies $0 < m < F_{n-1}$ then $\beta_{m+F_n} = \beta_{F_n}\beta_m$. In particular, if $n = \sum_{i=0}^{N} b_i F_i$, where $b_i b_{i-1} <_{lex} 11$, for all $i \in \{1, \dots, N\}$ then $\beta_n = \beta_{F_0}^{b_0} \dots \beta_{F_n}^{b_N}$.*

Proof. Let λ be an eigenvalue of S associated to the eigenvector $v = (v_i)_{i\geqslant 0} \in l^\infty(\mathbb{N})$. Since the transition probability from any nonnegative integer i to any integer $i+k$, with $k \geqslant 2$, is $p_{i,i+k} = 0$ (see proposition (1.8)), the operator S satisfies $S_{i,i+k} = 0$, for all $i, k \in \mathbb{N}$, with $k \geqslant 2$. Hence, for every integer $k \geqslant 1$, we have

$$\sum_{i=0}^{k} S_{k-1,i} v_i = \lambda v_{k-1}. \tag{3.1}$$

Then it is possible to prove by induction on k that for all integer $k \geqslant 1$, there exists a complex number $\beta_k = \beta_k(\overline{p}, \lambda) \in \mathbb{C}$ such that $v_k = \beta_k v_0$.

Put $k = F_n + j$, $j \in \{1, \ldots, m\}$, where $0 < m < F_{n-1}$. From proposition 1.10, we have that $S_{k-1,i} = 0$, for all $i \in \{0, \ldots, F_n - 1\}$. Hence, for all $j = 1, \ldots, m$, we have that

$$\sum_{i=F_n}^{F_n+j} S_{F_n+j-1,i} v_i = \lambda v_{F_n+j-1}.$$

From proposition 1.9 ($S_{i,j} = S_{i-F_n, j-F_n}$) we deduce that, for all $j = 1, \ldots, m$,

$$\lambda v_{F_n+j-1} = \sum_{i=F_n}^{F_n+j} S_{F_n+j-1,i} v_i = \sum_{i=F_n}^{F_n+j} S_{j-1,i-F_n} v_i = \sum_{l=0}^{j} S_{j-1,l} v_{l+F_n}. \tag{3.2}$$

Let $(w_i)_{0\leqslant i\leqslant j}$ defined by $w_i = v_{F_n+i}$. Hence, $\sum_{i=0}^{j} S_{j-1,i} w_i = \lambda w_{j-1}$, for $j \in \{1, \ldots, m\}$.

Therefore, $w_m = \beta_m w_0$, i.e. $v_{m+F_n} = \beta_m v_{F_n}$. Since $v_{F_n} = \beta_{F_n} v_0$ and $v_{F_n+m} = \beta_{F_n+m} v_0$, it follows that $\beta_{F_n+m} = \beta_m \beta_{F_n}$. □

Proof of the theorem 3.3: Let λ be an eigenvalue of S associated to the eigenvector $v = (v_i)_{i\geqslant 0} \in l^\infty(\mathbb{N})$. Hence, from lemma 3.5 we have that for all nonnegative integer $k \geqslant 1$, there exists a complex number $\beta_k = \beta_k(\overline{p}, \lambda) \in \mathbb{C}$ such that

$$v_k = \beta_k v_0. \tag{3.3}$$

Claim: $\beta_k = q_k$, for all $k \geqslant 1$.

Put $k = F_{2n-1}$ on relation (3.1). Using the fact that $F_{2n-1} - 1 = \underbrace{1010\ldots10}_{2n-2}1 = (10)^{n-1}1$ and the item (iv) from proposition 1.8, we obtain, for all nonnegative integer $n \geqslant 1$,

$$(1-p_1)v_{F_{2n-1}-1} + \sum_{i=1}^{n} p_1 p_2 \ldots p_i (1-p_{i+1}) v_{F_{2n-1}-F_{2i-1}} + p_1 p_2 \ldots p_{n+1} v_{F_{2n-1}} = \lambda v_{F_{2n-1}-1},$$

i.e.

$$v_{F_{2n-1}} = \frac{-1}{p_1 \ldots p_{n+1}} \left((1-p_1-\lambda) v_{F_{2n-1}-1} + \sum_{i=1}^{n} p_1 \ldots p_i (1-p_{i+1}) v_{F_{2n-1}-F_{2i-1}} \right). \tag{3.4}$$

From relation (3.4), changing n by $n+1$, we obtain

$$v_{F_{2n+1}} = \frac{-1}{p_1 \ldots p_{n+2}} \left((1-p_1-\lambda) v_{F_{2n+1}-1} + \sum_{i=1}^{n+1} p_1 \ldots p_i (1-p_{i+1}) v_{F_{2n+1}-F_{2i-1}} \right). \tag{3.5}$$

Hence, we have $v_{F_{2n+1}} = \left(1 - \frac{1}{p_{n+2}}\right) v_0 + \frac{1}{p_{n+2}} A_{F_{2n+1}}$, where $A_{F_{2n+1}}$ is given by:

$$\frac{-1}{p_1 \cdots p_{n+1}} \left((1-p_1-\lambda) v_{F_{2n-1}+F_{2n}-1} + \sum_{i=1}^{n} p_1 \ldots p_i (1-p_{i+1}) v_{F_{2n-1}+F_{2n}-F_{2i-1}} \right).$$

From relation (3.3) and lemma 3.5, we have

$$v_{F_{2n-1}+F_{2n}-1} = \beta_{F_{2n-1}+F_{2n}-1} v_0 = \beta_{F_{2n-1}-1} \beta_{F_{2n}} v_0 = \beta_{F_{2n}} v_{F_{2n-1}-1}.$$

We also obtain $v_{F_{2n-1}+F_{2n}-F_{2i-1}} = \beta_{F_{2n}} v_{F_{2n-1}-F_{2i-1}}$, for all $i \in \{1, \ldots, n\}$.
Hence, from relations (3.4) and (3.5) we obtain $A_{F_{2n+1}} = \beta_{F_{2n}} v_{F_{2n-1}} = \beta_{F_{2n}} \beta_{F_{2n-1}} v_0$.
Therefore, we conclude that $\beta_{F_{2n+1}} = \frac{1}{p_{n+2}} \beta_{F_{2n}} \beta_{F_{2n-1}} - \left(\frac{1}{p_{n+2}} - 1 \right)$, for all $n \geqslant 1$.
The case $k = F_{2n}$ can be done by the same way.
On the other hand, it's easy to check that $v_{F_0} = q_{F_0} v_0$ and $v_{F_1} = q_{F_1} v_0$, i.e.

$$\beta_{F_0} = -\frac{1-\lambda-p_1}{p_1} = q_{F_0} \quad \text{and} \quad \beta_{F_1} = \frac{1}{p_2} q_{F_0}^2 - \left(\frac{1}{p_2} - 1 \right) = q_{F_1}.$$

Therefore, $\beta_n = q_n$, for all $n \geqslant 1$.
Hence, we have $\sigma_{pt}(S) = \{\lambda \in \mathbb{C} : (q_n(\lambda))_{n \in \mathbb{N}} \text{ is bounded}\}$ and we are done. □

Conjecture 3.6. *There exists* $0 < d < 1$ *such that if* $p_i > d$*, for all* $i \geqslant 1$ *then* $\sigma_{pt}(S) = E$.

Remark 3.7. In [CM], we prove a particular case of conjecture 3.6. Particulary we prove that if $p_i = p$, for all $i \geqslant 1$, and $\frac{-1+\sqrt{5}}{2} < p < 1$ then $E \cap \mathbb{R} = \sigma_{pt}(S) \cap \mathbb{R}$.

Theorem 3.8. *In* $l^\infty(\mathbb{N})$*, the set* E *is contained in* $\sigma_a(S)$*. In particular* $E \subset \sigma(S)$.

Proof. Let $\lambda \in E$ and suppose that $\lambda \notin \sigma_{pt}(S)$. We will prove that $\lambda \in \sigma_a(S)$. In fact, for each $k \geqslant 2$, consider

$$w^{(k)} = (w_0^k, w_1^k, w_2^k, \ldots, w_k^k, w_{k+1}^k, w_{k+2}^k, \ldots)^t = (1, q_1(\lambda), q_2(\lambda), \ldots, q_k(\lambda), 0, 0, \ldots)^t,$$

where $(q_n(\lambda))_{n \geqslant 1} = (q_n)_{n \geqslant 1}$ is the sequence defined before. Define $u^{(k)} := \frac{w^{(k)}}{\|w^{(k)}\|_\infty}$.
Claim: $\lim\limits_{n \to +\infty} \|(S - \lambda I) u^{(F_n)}\|_\infty = 0$.
In fact, for all $i \in \{0, \ldots, k-1\}$, we have $((S-\lambda I)u^{(k)})_i = 0$ and $u_i = 0$, for all $i > k$. Hence, note that

$$\|(S-\lambda I)u^{(k)}\|_\infty = \sup_{i \geqslant 0} \left| \sum_{j=0}^{+\infty} (S-\lambda I)_{ij} u_j^{(k)} \right| = \sup_{i \geqslant k} \left\{ \frac{\left| \sum_{j=0}^{k} (S-\lambda I)_{ij} w_j^{(k)} \right|}{\|w^{(k)}\|_\infty} \right\}.$$

If $k = F_n$ then, for $i \geqslant k = F_n$, we have:
$a)$ If $i = F_n$ then $S_{i,j} = 0$, for all $j \in \{0, \ldots, F_n - 1\}$, and $S_{F_n, F_n} = 1 - p_1$. Therefore, $\left| \sum_{j=0}^{F_n} (S-\lambda I)_{ij} w_j^{(F_n)} \right| = |1 - p_1 - \lambda| |q_{F_n}|$.
$b)$ If $F_n < i < F_{n+1} - 1$ then $S_{i,j} = 0$, for all $j \in \{0, \ldots, F_n - 1\}$, and $S_{i,j} \leqslant p_1$, for $j = F_n$. Therefore, $\left| \sum_{j=0}^{F_n} (S-\lambda I)_{ij} w_j^{(F_n)} \right| \leqslant p_1 |q_{F_n}|$.
$c)$ If $i = F_{n+1} - 1$ then $S_{i,j} = 0$, for all $j \in \{1, \ldots, F_n - 1\}$, and $S_{i,j} \leqslant p_1$, for $j = 0, F_n$. Therefore, $\left| \sum_{j=0}^{F_n} (S-\lambda I)_{ij} w_j^{(F_n)} \right| \leqslant p_1 + p_1 |q_{F_n}|$.

d) If $i \geqslant F_{n+1}$ then $S_{i,j} = 0$, for all $j \in \{1, \ldots, F_n\}$, and $S_{i,j} \leqslant p_1$, for $j = 0$. Therefore, $\left|\sum_{j=0}^{F_n}(S-\lambda I)_{ij} w_j^{(F_n)}\right| \leqslant p_1$.

Hence, from a), b), c) and d) it follows that

$$\|(S-\lambda I)u^{(F_n)}\|_\infty \leqslant \frac{|1-p_1-\lambda||q_{F_n}| + p_1|q_{F_n}| + p_1}{\|w^{(F_n)}\|_\infty}. \tag{3.6}$$

Since $\lambda \in E$ and $\lambda \notin \sigma_{pt}(S)$ if follows that $(q_{F_n})_{n\geqslant 0}$ is a bounded sequence and $(q_n)_{n\geqslant 0}$ is not. Therefore, we have $\lim_{n\to+\infty}\|w^{(F_n)}\|_\infty = +\infty$, which implies from relation 3.6 that $\lim\limits_{n\to+\infty}\|(S-\lambda I)u^{(F_n)}\|_\infty = 0$. Therefore, $\lambda \in \sigma_a(S) \subset \sigma(S)$. ☐

Conjecture 3.9. *$E = \sigma_a(S)$.*

3.1. **Generalization.** Let $d > 2$ a integer number and let us consider the sequence $(F_n)_{n\geqslant 0}$ given by $F_n = a_1F_{n-1} + \ldots + a_dF_{n-d}$, for all $n \geqslant d$, with initial conditions $F_0 = 1$ and $F_n = a_1F_{n-1} + \ldots + a_nF_0 + 1$, for all $n \in \{1, \ldots, d-1\}$, where a_i are nonnegative integers, for $i \in \{1, \ldots, d\}$, satisfying $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_d \geqslant 1$.

By using the greedy algorithm we can write every nonnegative integer N, in a unique way, as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N)F_i$ where the digits $\varepsilon_j(N)$ satisfy the relation $\varepsilon_i\varepsilon_{i-1}\ldots\varepsilon_{i-d+1} <_{lex} a_1a_2\ldots a_d$, for all $i \geqslant d-1$.

It is known that the addition of 1 in base $(F_n)_{n\geqslant 0}$ is given by a finite transducer. By using the same construction as was done in Fibonacci base, we can define the stochastic adding machine associated to the sequences $(F_n)_{n\geqslant 0}$ and $\overline{p} = (p_i)_{i\geqslant 1}$ and we can also prove that $\sigma_{pt}(S_{\overline{p}}) = \{\lambda \in \mathbb{C} : (q_n(\lambda))_{n\geqslant 0} \text{ is bounded}\}$, where $q_{F_i}(\lambda)$ are polynomials fixed in λ for all $i \in \{0, \ldots, d-1\}$,

$$q_{F_{dn+i}}(\lambda) = \tfrac{1}{p_{n+1+i}} q_{F_{dn+i-1}}^{a_1}(\lambda) q_{F_{dn+i-2}}^{a_2}(\lambda) \ldots q_{F_{dn+i-d}}^{a_d}(\lambda) - \left(\tfrac{1}{p_{n+1+i}} - 1\right)$$

for all $n \geqslant 1$ and for each $i \in \{0, 1, \ldots, d-1\}$, and $q_N(\lambda) = q_{F_0}^{\varepsilon_0}(\lambda)\ldots q_{F_n}^{\varepsilon_n}(\lambda)$, where $N = \sum_{i=0}^n \varepsilon_i F_i$.

In particular, if $a_1 = a_2 = \ldots = a_d = 1$ then $q_{F_0}(\lambda) = \frac{\lambda-(1-p_1)}{p_1}$ and $q_{F_i}(\lambda) = \frac{1}{p_{i+1}}(q_{F_{i-1}}(\lambda))^2 - \left(\frac{1}{p_{i+1}} - 1\right)$, for all $i \in \{1, \ldots, d-1\}$.

4. Topological properties of the stochastic Fibonacci adding machine

Theorem 4.1. *Suppose there exists $\delta > 0$ such that $p_i > \delta$, for all $i \geqslant 0$. Then E is compact and $\mathbb{C} \setminus E$ is connected.*

In definition 3.2, we have that for each $n \in \mathbb{N}$, $q_{F_n} : \mathbb{C} \longrightarrow \mathbb{C}$ are maps defined by $q_{F_0}(z) = \frac{z-(1-p_1)}{p_1}$, $q_{F_1}(z) = \frac{1}{p_2}(q_{F_0}(z))^2 - \left(\frac{1}{p_2} - 1\right)$ and $q_{F_n}(z) = \frac{1}{r_n} q_{F_{n-1}}(z) q_{F_{n-2}}(z) - \left(\frac{1}{r_n} - 1\right)$, where $r_n = p_{[\frac{n+1}{2}]+1}$, for all $n \geqslant 1$ and the set E is defined by $E := \{z \in \mathbb{C} : (q_{F_n}(z))_{n\geqslant 0} \text{ is bounded}\}$.

Before to proof the theorem 4.1, it will be necessary the followings results:

Lemma 4.2. *If there exists a nonnegative integer k such that $|q_{F_n}(z)| > 1$ for $n \in \{k, k+1\}$ then the sequence $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded.*

Proof. Since $|q_{F_n}(z)| > 1$ for $n \in \{k, k+1\}$ then there exists a real number $A > 1$ such that $|q_{F_n}(z)| > A$ for $n \in \{k, k+1\}$. Hence, we have

$$|q_{F_{k+2}}(z)| \geqslant \frac{1}{r_{k+2}}|q_{F_{k+1}}(z)q_{F_k}(z)| - |\frac{1}{r_{k+2}} - | > \frac{A^2}{r_{k+2}} - \frac{1}{r_{k+2}} + 1, \text{ i.e. } |f_{k+2}(z)| > A^2$$

and likewise we have $|q_{F_{k+3}}(z)| > A^3$ and $|q_{F_{k+4}}(z)| > A^5$.

Continuing in this way, we conclude that $|q_{F_{k+l}}(z)| > A^{F_{l-1}}$, for all $l \in \mathbb{N}$, $l \geqslant 2$.

Since $A > 1$, it follows that $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded. □

Proposition 4.3. *Suppose there exists $\delta > 0$ such that $p_i > \delta$, for all $i \geqslant 1$. Then there exists $R > 1$ such that, if $|q_{F_k}(z)| > R$ for some $k \in \mathbb{N}$ then $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded.*

Proof. Let $R > 1$ be a real number such that $R > \frac{2}{\delta} - 1$. Hence, we have that

$$\frac{2}{p_2} - 1 < R < R\delta\left(R + 1 - \frac{1}{\delta}\right). \tag{4.1}$$

If $|q_{F_0}(z)| > R$ then $|q_{F_1}(z)| \geqslant \frac{1}{p_2}|q_{F_0}(z)|^2 - \left|\frac{1}{p_2} - 1\right| > \frac{1}{p_2}R^2 - \left(\frac{1}{p_2} - 1\right)$, i.e. $|q_{F_1}(z)| > R^2$. Therefore, it follows from lemma 4.2 that the sequence $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded.

If $|q_{F_1}(z)| > R$ then $R < |q_{F_1}(z)| \leqslant \frac{1}{p_2}|(q_{F_0}(z))^2| + \left|\frac{1}{p_2} - 1\right| = \frac{1}{p_2}|q_{F_0}(z)|^2 + \frac{1}{p_2} - 1$ and it implies $|q_{F_0}(z)|^2 > p_2\left(R + 1 - \frac{1}{p_2}\right) > \delta\left(R + 1 - \frac{1}{\delta}\right) > 1$, where the last inequality follows from relation (4.1). Hence, $|q_{F_0}(z)| > 1$ and from lemma 4.2 we have that the sequence $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded.

By induction on k, suppose that for all $i \in \{0, 1, \ldots, k-1\}$ we have: if $|q_{F_i}(z)| > R$ then $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded.

Suppose that $|q_{F_k}(z)| > R$. If $|q_{F_{k+1}}(z)| > 1$, it follows from lemma 4.2 that $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded. Therefore, suppose $|q_{F_{k+1}}(z)| \leqslant 1$. Then we have

$$1 \geqslant |q_{F_{k+1}}(z)| \geqslant \frac{1}{r_{k+1}}|q_{F_k}(z)||q_{F_{k-1}}(z)| - \left|\frac{1}{r_{k+1}} - 1\right| > \frac{R}{r_{k+1}}|q_{F_{k-1}}(z)| - \left(\frac{1}{r_{k+1}} - 1\right),$$

i.e. $\frac{1}{R} > |f_{k-1}(z)|$. On the other hand, we have that

$$R < |q_{F_k}(z)| \leqslant \frac{1}{r_k}|q_{F_{k-1}}(z)||q_{F_{k-2}}(z)| + \left|\frac{1}{r_k} - 1\right| < \frac{1}{r_k}\frac{1}{R}|q_{F_{k-2}}(z)| + \frac{1}{r_k} - 1,$$

i.e. $|q_{F_{k-2}}(z)| > Rr_k\left(R + 1 - \frac{1}{r_k}\right) > R\delta\left(R + 1 - \frac{1}{\delta}\right) > R$. Then by induction hypothesis we have that the sequence $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded. □

Proposition 4.4. *Suppose there exists $\delta > 0$ such that $p_i > \delta$, for all $i \geqslant 1$. Then there exists a real number $R > 1$ such that $E = \bigcap_{n=0}^{+\infty} q_{F_n}^{-1}\overline{D(0,R)}$, where $\overline{D(0,R)}$ is the closed disk of centre 0 and radius R.*

Proof. It is a direct consequence of proposition 4.3. □

Remark 4.5. From proof of proposition 4.3, we also have $|q_{F_k}(z)| \geqslant R$ for some nonnegative integer k implies $(q_{F_n}(z))_{n\geqslant 0}$ is not bounded. Hence, $E = \bigcap_{n=0}^{+\infty} q_{F_n}^{-1}D(0,R)$.

Proof of theorem 4.1: It follows directly from proposition 4.4 that E is compact.

From proposition 4.3, there exists $R > 0$ such that $\mathbb{C} \setminus E = \bigcup_{n=0}^{+\infty} \mathbb{C} \setminus q_{F_n}^{-1}\overline{D(0,R)}$. Hence, since $\mathbb{C} \setminus \overline{D(0,R)}$ is connected, it follows from *maximum modulus principle* that for each

holomorphic map q_{F_n}, $\mathbb{C} \setminus q_{F_n}^{-1}\overline{D(0,R)}$ is connected for all $n \geqslant 0$. On the other hand, since $\mathbb{C} \setminus q_{F_n}^{-1}\overline{D(0,R)}$ contains a neighbourhood of infinity for all $n \geqslant 0$, we deduce that

$$\mathbb{C} \setminus E = \bigcup_{n=0}^{+\infty} \mathbb{C} \setminus q_{F_n}^{-1}\overline{D(0,R)} \text{ is connected.}$$

□

Conjecture 4.6. *E is a compact set and $\mathbb{C}\setminus E$ is a connected set, even supposing* $\liminf_{i\to+\infty} p_i = 0$.

Proposition 4.7. *Suppose there exists $\delta > 0$ such that $p_i > \delta$, for all $i \geqslant 1$. Then there exists $R > 1$ such that $E = \bigcap_{n=0}^{+\infty} q_{F_n}^{-1} D(0,R)$ and $q_{F_{n+1}}^{-1} D(0,R) \subset q_{F_n}^{-1} D(0,R)$, for all $n \geqslant 0$.*

Proof. Let R be a real number such that $R > \frac{2}{\delta} - 1$. From remark 4.5, we have that $E = \bigcap_{n=0}^{+\infty} q_{F_n}^{-1} D(0,R)$.

We will prove by induction on k that $q_{F_{k+1}}^{-1} D(0,R) \subset q_{F_k}^{-1} D(0,R)$, for all $k \geqslant 0$.

In fact, suppose that $q_{F_{k+1}}^{-1} D(0,R) \subset q_{F_k}^{-1} D(0,R)$, for all $k \in \{0,\dots,n-1\}$, with $n \geqslant 4$. Let $z \in q_{F_{n+1}}^{-1} D(0,R)$ and suppose that $q_{F_n}(z) \geqslant R$.

Let $A = 1 - \frac{1}{\delta}$. Hence, $\left|1 - \frac{1}{r_n}\right| < |A|$ for all $n \geqslant 1$ and

$$R > |q_{F_{n+1}}(z)| \geqslant \tfrac{1}{r_{n+1}} R |q_{F_{n-1}}(z)| - \left|\tfrac{1}{r_{n+1}} - 1\right|, \text{ i.e. } |q_{F_{n-1}}(z)| < \tfrac{r_{n+1}(R+|A|)}{R} = \mathcal{O}(1).$$

Therefore, by induction hypothesis we have that

$$|q_{F_k}(z)| < R \text{ for all } k \in \{1,\dots,n-2\}. \tag{4.2}$$

On the other hand

$$|q_{F_n}(z)| \leqslant \tfrac{1}{r_n}|q_{F_{n-1}}(z)||q_{F_{n-2}}(z)| + \left|\tfrac{1}{r_n} - 1\right| < \tfrac{1}{r_n}\mathcal{O}(1)|q_{F_{n-2}}(z)| + |A|, \text{ i.e.}$$
$$|q_{F_{n-2}}(z)| > r_n \tfrac{|q_{F_n}(z)| - |A|}{\mathcal{O}(1)} > \delta \frac{R - |A|}{\mathcal{O}(1)} = \mathcal{O}(R).$$

Thus, continuing this way we have that $|q_{F_{n-3}}(z)| < \mathcal{O}\left(\frac{1}{R}\right)$ and $|q_{F_{n-4}}(z)| > \mathcal{O}(R^2)$. Choosing R large enough, we have $|q_{F_{n-4}}(z)| > R$ and this contradicts the relation (4.2). Therefore, $z \in q_{F_n}^{-1}(D(0,R))$.

Now, in order to finish the proof of this theorem, we will prove that $q_{F_{k+1}}^{-1} D(0,R) \subset q_{F_k}^{-1} D(0,R)$, for $k = 0,1,2,3$.

Case k=0: Let $z \in q_{F_1}^{-1} D(0,R)$. Then $\frac{1}{p_2}|q_{F_0}(z)|^2 - \left(\frac{1}{p_2} - 1\right) < R$. Therefore,

$$\tfrac{1}{p_2}(|q_{F_0}(z)|^2 - 1) < R - 1 \text{ and } |q_{F_0}(z)| < \sqrt{R} < R, \text{ i.e. } z \in q_{F_0}^{-1} D(0,R).$$

Case k=1: Let $z \in q_{F_2}^{-1} D(0,R)$. Then $\frac{1}{r_2}|q_{F_1}(z)||q_{F_0}(z)| - \left(\frac{1}{r_2} - 1\right) < R$. Therefore, $|q_{F_1}(z)||q_{F_0}(z)| < r_2 R + 1 - r_2 < R$. Thus, we have that $|q_{F_1}(z)| < (r_2 R + 1 - r_2)^{\frac{3}{4}}$ or $|q_{F_0}(z)| < (r_2 R + 1 - r_2)^{\frac{1}{4}}$.

If $|q_{F_1}(z)| < (r_2 R + 1 - r_2)^{\frac{3}{4}} < R$ then $z \in q_{F_1}^{-1} D(0,R)$.

If $|q_{F_0}(z)| < (r_2 R + 1 - r_2)^{\frac{1}{4}}$ then

$$|q_{F_1}(z)| \leqslant \frac{1}{r_2}|q_{F_0}(z)|^2 + \left|\frac{1}{r_2} - 1\right| \leqslant \frac{1}{r_2}(r_2R + 1 - r_2)^{\frac{1}{2}} + \left|\frac{1}{r_2} - 1\right|.$$

Thus, choosing R large enough such that $\frac{1}{r_2}(r_2R + 1 - r_2)^{\frac{1}{2}} + \left|\frac{1}{r_2} - 1\right| < R$, we have $z \in q_{F_1}^{-1}D(0,R)$.

Case k=2: Let $z \in q_{F_3}^{-1}D(0,R)$. Then $\frac{1}{r_3}|q_{F_2}(z)||q_{F_1}(z)| - \left(\frac{1}{r_3} - 1\right) < R$. Therefore, $|q_{F_2}(z)||q_{F_1}(z)| < r_3R + 1 - r_3 \leqslant R$. Thus, we have that $|q_{F_2}(z)| < (r_3R + 1 - r_3)^{\frac{1}{2}}$ or $|q_{F_1}(z)| < (r_3R + 1 - r_3)^{\frac{1}{2}}$.

If $|q_{F_2}(z)| < (r_3R + 1 - r_3)^{\frac{1}{2}} \leqslant R$ then $z \in q_{F_2}^{-1}D(0,R)$.

If $|q_{F_1}(z)| < (r_3R+1-r_3)^{\frac{1}{2}}$ then $\frac{1}{p_2}|q_{F_0}(z)|^2 - \left(\frac{1}{p_2} - 1\right) < (r_3R+1-r_3)^{\frac{1}{2}}$, i.e. $|q_{F_0}(z)| < \sqrt{p_2}\left((r_3R + 1 - r_3)^{\frac{1}{2}} + \left(\frac{1}{p_2} - 1\right)\right)^{\frac{1}{2}}$. Hence, $|q_{F_2}(z)| \leqslant \frac{1}{p_2}|q_{F_1}(z)||q_{F_0}(z)| + \left(\frac{1}{p_2} - 1\right)$, i.e. $|q_{F_2}(z)| < \frac{1}{p_2}(r_3R + 1 - r_3)^{\frac{1}{2}}\sqrt{p_2}\left((r_3R + 1 - r_3)^{\frac{1}{2}} + \left(\frac{1}{p_2} - 1\right)\right)^{\frac{1}{2}} + \left(\frac{1}{p_2} - 1\right) = \mathcal{O}(R^{\frac{3}{4}})$.

Thus, choosing R large enough, we have $|q_{F_2}(z)| < R$, i.e. $z \in q_{F_2}^{-1}D(0,R)$.

Case k=3 This case can be done by the same way. □

Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be a non-null polynomial and define the set $\mathcal{F}_h := \{z \in \mathbb{C} : (g_n \circ \ldots \circ g_2(h(z), z))_{n\geqslant 2}$ is bounded$\}$, where $g_n : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is defined in the remark 3.4.

Remark 4.8. If $h(z) = \frac{1}{p_2}z^2 - \left(\frac{1}{p_2} - 1\right)$ then E and $\mathcal{F}_h$ are isomorphic by l, where $l : \mathbb{C} \longrightarrow \mathbb{C}$ is defined by $l(\lambda) = \frac{1}{p_1}\lambda - \left(\frac{1}{p_1} - 1\right)$.

Claim 4.9. The above results are true, if we consider the set $\mathcal{F}_h$, for all non-null polynomial h. In particular, if we suppose there exists $\delta > 0$ such that $p_i > \delta$, for all $i \geqslant 1$ then we have that $\mathbb{C} \setminus \mathcal{F}_h$ is a connected set and there exists $R > 1$ such that $\mathcal{F}_h = \bigcap_{n=0}^{\infty} \varphi_n^{-1}D(0,R)$, where $\varphi_0(z) = z$, $\varphi_1(z) = h(z)$ and $\varphi_n(z) = \frac{1}{r_n}\varphi_{n-1}(z)\varphi_{n-2}(z) - \left(\frac{1}{r_n} - 1\right)$, for all $n \geqslant 2$. Furthermore we have that $\varphi_{n+1}^{-1}D(0,R) \subset \varphi_n^{-1}D(0,R)$, for all $n \in \mathbb{N}$. .

Theorem 4.10. *Let $h(z) = a_2z^2 + a_3z^3 + \ldots + a_nz^n$, with $n \geqslant 2$ and $a_2, \ldots, a_n \in \mathbb{C}$, and suppose that $\liminf_{i\to+\infty} p_i > 0$. Then we have the followings results:*

a) If $p_3 < \frac{1}{2}$ and $p_2 = 1$ or $0 < p_2 < 1 - p_3$ then $\mathcal{F}_h$ is a non-connected set.

b) If $p_i = 1$ for all $i \in \{2, 3, \ldots, k\}$ and $p_{k+1} < \frac{1}{2}$, for some $k \geqslant 3$ then $\mathcal{F}_h$ is a non-connected set.

Before to prove the theorem 4.10, it will be necessary the following lemma which is a particular case of the Riemann-Hurwitz formula (see the theorem 7.2 of the page 70 in [M] and the theorem 1.1.4 and the lemma 1.1.5 of the page 10 in [MNTU]).

Lemma 4.11. *Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial function and $R > 0$. Then*

$$\varphi^{-1}D(0,R) \text{ is connected if only if } \{z \in \mathbb{C} : \varphi'(z) = 0\} \subset \varphi^{-1}D(0,R).$$

Proof of theorem 4.10:

a) If $p_2 = 1$ and $p_3 < \frac{1}{2}$ then $\varphi_0(0) = \varphi_1(0) = \varphi_2(0) = 0$ and $|\varphi_3(0)| = |\varphi_4(0)| = \frac{1}{p_3} - 1 > 1$, which implies from lemma 4.2 that $(\varphi_n(0))_{n\geqslant 0}$ is not bounded.

If $p_3 < \frac{1}{2}$ and $0 < p_2 < 1-p_3$ then $\varphi_0(0) = \varphi_1(0) = 0$, $\varphi_2(0) = 1-\frac{1}{p_2}$, $|\varphi_3(0)| = \frac{1}{p_3}-1 > 1$. Furthermore, $p_2 < 1 - p_3$ implies $\varphi_4(0) = \frac{1}{p_3}\left(1 - \frac{1}{p_3}\right)\left(1 - \frac{1}{p_2}\right) - \left(\frac{1}{p_3} - 1\right) > 1$.
Thus, since $|\varphi_3(0)| > 1$ and $|\varphi_4(0)| > 1$, it follows from lemma 4.2 that $(\varphi_n(0))_n$ is not bounded.

Therefore, in both cases we have that $0 \notin \mathcal{F}_h$.

Hence, there exists a integer number $n \geqslant 1$ such that $0 \notin \varphi_k^{-1}D(0,R)$, for all $k \geqslant n$, where R is the real number defined at remark 4.9.

Since $\varphi_n'(0) = 0$, for all $n \geqslant 1$, we deduce from lemma 4.11 that $\varphi_k^{-1}D(0,R)$ is not connected for all $k \geqslant n$. Since $\mathcal{F}_h = \bigcap_{n=0}^{\infty} \varphi_n^{-1}D(0,R)$ and $\varphi_{n+1}^{-1}D(0,R) \subset \varphi_n^{-1}D(0,R)$, for all $n \in \mathbb{N}$, it follows that $\mathcal{F}_h$ is not connected.
$b)$ Let $k \geqslant 3$ and suppose that $p_i = 1$ for all $i \in \{2,3,\ldots,k\}$ and $p_{k+1} < \frac{1}{2}$. Hence, we have that $\varphi_0(0) = \varphi_1(0) = \ldots = \varphi_{2k-2}(0) = 0$ and $|\varphi_{2k-1}(0)| = |\varphi_{2k}(0)| = \frac{1}{p_{k+1}} - 1 > 1$. Thus, in the same way that was done in item $a)$, it follows that $\mathcal{F}_h$ is not connected.

□

Corollary 4.12. *Suppose that* $\liminf\limits_{i\to+\infty} p_i > 0$. *If* $p_i = 1$ *for all* $i \in \{2,3,\ldots,k\}$ *and* $p_{k+1} < \frac{1}{2}$, *for some* $k \geqslant 2$ *then* E *is a non-connected set.*

Proof. Let $h(z) = \frac{1}{p_2}z^2 - \left(\frac{1}{p_2} - 1\right) = z^2$. From theorem 4.10 we have that $\mathcal{F}_h$ is not connected. Therefore, from remark 4.8 it follows that E is a non-connected set. □

Conjecture 4.13. *There exists* $0 < \delta < 1$ *such that if* $p_i > \delta$, *for all* $i \geqslant 1$ *then* E *is connected.*

The conjecture 4.13 is motivated by the fact that the authors in [ABMS] proved that there exits $0 < a < 1$ such that if $p_i = p > a$, for all $i \geqslant 1$ then E is quasi-disk.

We can see some possibilities for the set E in the following figures:

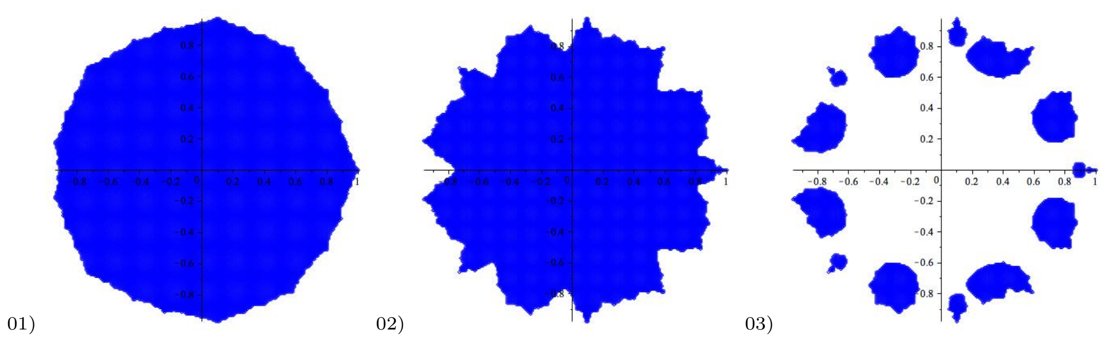
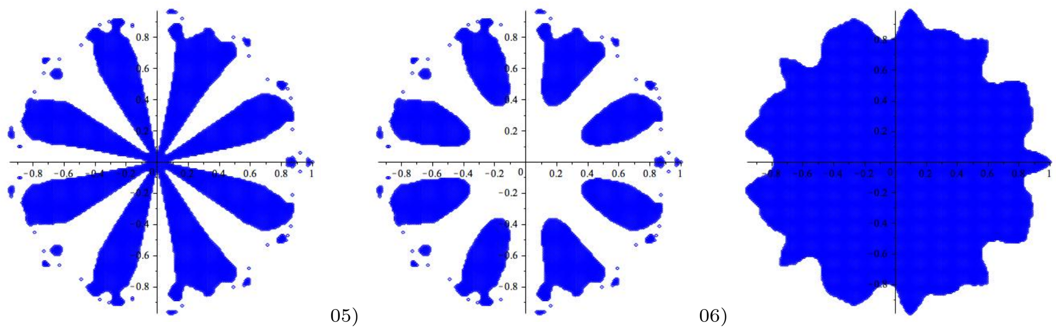
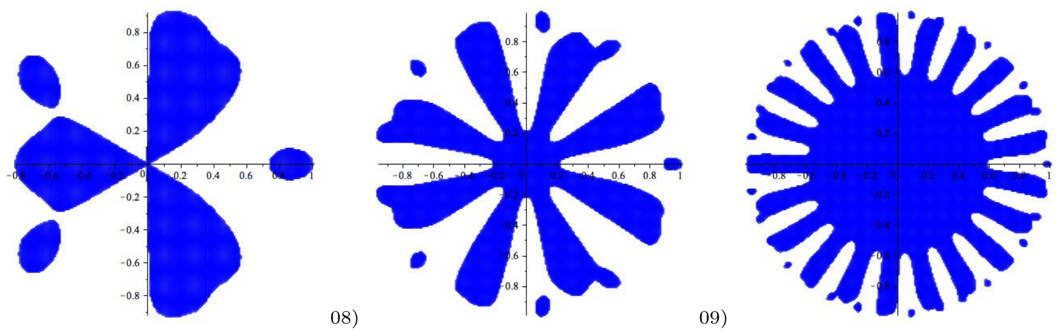


FIGURE 4. 01) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 0,909$, $p_4 = 0,833$, $p_5 = 0,769$, $p_6 = 0,714$, $p_7 = 0,666$, $p_8 = 0,625$, $p_9 = 0,588$, $p_{10} = 0,555$, $p_{11} = 0,526$, $p_i = 1$, for all $i \in \{12,\ldots,17\}$. 02) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,625$, $p_5 = 0,588$, $p_6 = 0,625$, $p_7 = 0,666$, $p_8 = 0,714$, $p_9 = 0,769$, $p_{10} = 0,833$, $p_{11} = 0,909$, $p_i = 1$, for all $i \in \{12,\ldots,17\}$. 03) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,588$, $p_5 = 0,588$, $p_6 = 0,625$, $p_7 = 0,666$, $p_8 = 0,714$, $p_9 = 0,769$, $p_{10} = 0,833$, $p_{11} = 0,909$, $p_i = 1$, for all $i \in \{12,\ldots,17\}$.



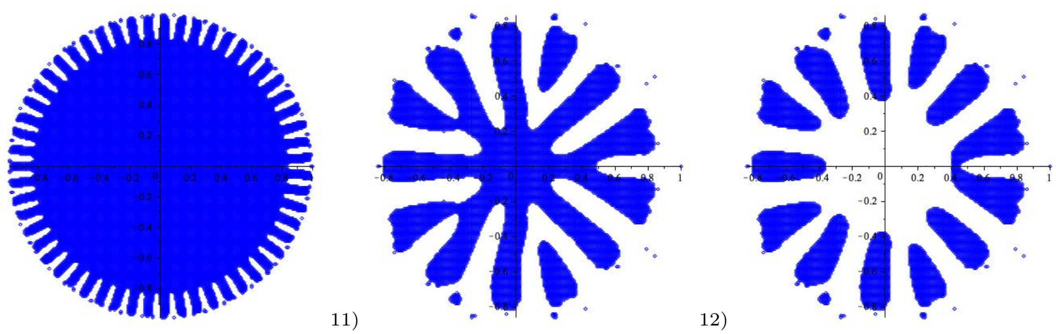
04) 05) 06)

FIGURE 5. 04) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,625$, $p_5 = 0,714$, $p_6 = 0,403637066$, $p_7 = 0,833$, $p_8 = 0,833$, $p_9 = 909$, $p_i = 1$, for all $i \in \{10,\dots,17\}$. 05) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,625$, $p_5 = 0,714$, $p_6 = 0,403$, $p_7 = 0,833$, $p_8 = 0,833$, $p_9 = 909$, $p_i = 1$, for all $i \in \{10,\dots,17\}$. 06) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,588$, $p_5 = 0,769$, $p_6 = 0,833$, $p_7 = 0,909$, $p_8 = 0,833$, $p_i = 1$, for all $i \in \{9,\dots,17\}$.



07) 08) 09)

FIGURE 6. 07) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 0,5$, $p_i = 1$, for all $i \in \{4,\dots,17\}$. 08) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,5$, $p_i = 1$, for all $i \in \{5,\dots,17\}$. 09) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 1$, $p_5 = 0,5$, $p_i = 1$, for all $i \in \{6,\dots,17\}$.



10) 11) 12)

FIGURE 7. 10) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 1$, $p_5 = 1$, $p_6 = 0,5$ $p_i = 1$, for all $i \in \{7,\dots,17\}$. 11) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,625$, $p_5 = 0,284864$, $p_6 = 0,625$, $p_7 = 0,714$, $p_8 = 0,769$, $p_9 = 0,833$, $p_{10} = 0,909$, $p_i = 1$, for all $i \in \{11,\dots,17\}$. 12) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,625$, $p_5 = 0,284859$, $p_6 = 0,625$, $p_7 = 0,714$, $p_8 = 0,769$, $p_9 = 0,833$, $p_{10} = 0,909$, $p_i = 1$, for all $i \in \{11,\dots,17\}$.

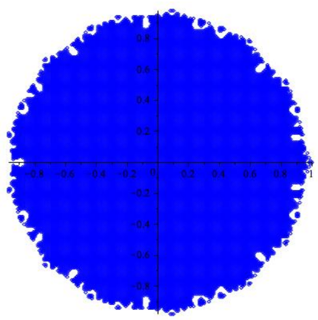
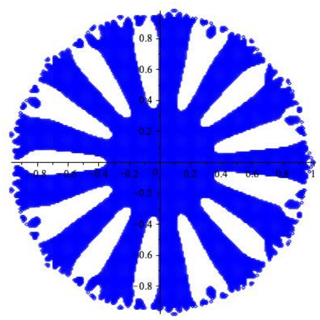
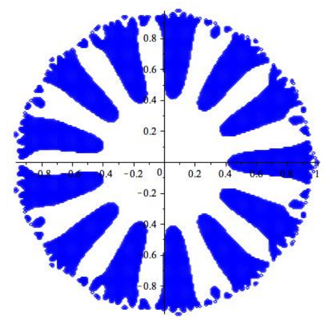

13)

14)

15)

FIGURE 8. 13) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,833$, $p_5 = 0,769$, $p_6 = 0,833$, $p_7 = 0,465$, $p_8 = 0,833$, $p_9 = 909$, $p_i = 1$, for all $i \in \{10,\dots,17\}$. 14) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,833$, $p_5 = 0,769$, $p_6 = 0,833$, $p_7 = 0,46041639$, $p_8 = 0,833$, $p_9 = 909$, $p_i = 1$, for all $i \in \{10,\dots,17\}$. 15) $p_1 = 1$, $p_2 = 0,999$, $p_3 = 1$, $p_4 = 0,833$, $p_5 = 0,769$, $p_6 = 0,833$, $p_7 = 0,46041617$, $p_8 = 0,833$, $p_9 = 909$, $p_i = 1$, for all $i \in \{10,\dots,17\}$.

Acknowledgments. I heartily thank *Ali Messaoudi* for the introduction to the subject and for the many helpful comments, discussions and suggestions. I want also to thank the financial supports from *FAPESP* grant $2010/19731-7$ and from *CNPq* grant $150311/2015-0$.

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